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Jürgen Franke · Wolfgang K. Härdle
Christian M. Hafner

# Statistics of Financial Markets

## An Introduction

Second Edition

Springer

Professor Dr. Jürgen Franke
University of Kaiserslautern
P.O. Box 3049
67653 Kaiserslautern
Germany
franke@mathematik.uni-kl.de

Professor Dr. Christian M. Hafner
Université catholique de Louvain
Institut de statistique
Voie du Roman Pays, 20
1348 Louvain-la-Neuve
Belgium
christian.hafner@uclouvain.be

Professor Dr. Wolfgang K. Härdle
Humboldt-Universität zu Berlin
CASE-Center for Applied
Statistics and Economics
Spandauer Straße 1
10178 Berlin
Germany
haerdle@wiwie.hu-berlin.de

*front cover figure:* The photo on the cover of the bull and bear in front of the Frankfurt Stock Exchange was taken by Professor Wolfgang Härdle

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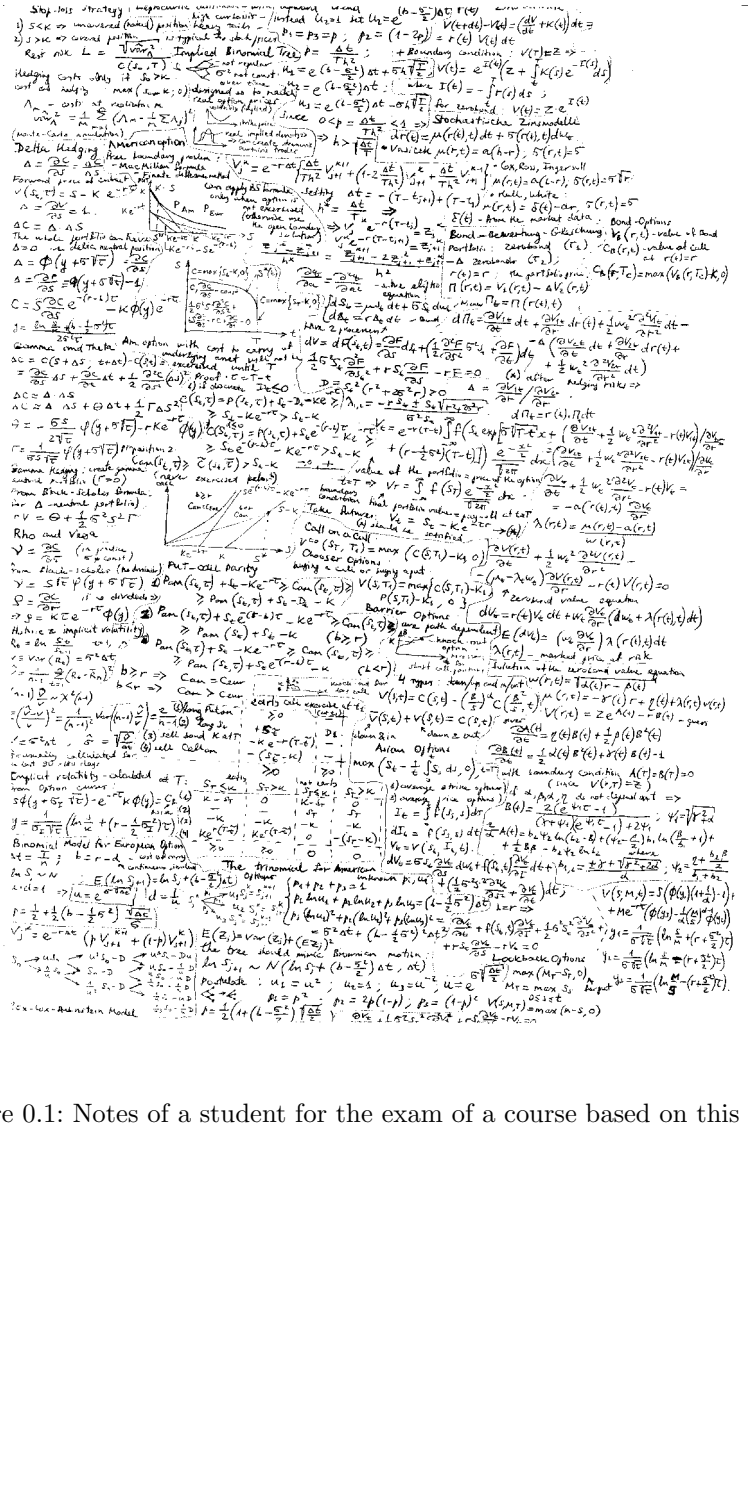

Figure 0.1: Notes of a student for the exam of a course based on this book.

Figure 0.2: Notes of a student for the exam of a course based on this book.

Figure 0.3: Notes of a student for the exam of a course based on this book.

Figure 0.4: Notes of a student for the exam of a course based on this book.

Figure 0.5: Notes of a student for the exam of a course based on this book.

# Preface to the Second Edition

After the success of the first edition we felt obliged to catch up with the rapidly growing literature in financial statistics and econometrics. This second edition expands on material that was only briefly covered in the previous edition. As an example, Chapter 17 on copula is an extensive update of the literature and describes some of our own research in this area. In the chapter on time series with stochastic volatility (Chapter 13), we present a critique of standard stationary GARCH modelling and describe an alternative nonparametric way of modelling based on the idea of a time-varying unconditional variance, and hence a non-stationary process. This new view of volatility modelling seems to provide promising results in prediction when compared with standard GARCH models. We have substantially augmented the section on risk management (Section 6.3), including the *Volga* and *Vanna* coefficients and the recent work on *realised volatility*. Another very active part of research is on multivariate GARCH models, where we provide an updated review in Section 13.4. We have included a new section on simulation techniques and an entire chapter on Credit Risk Management. In addition to these changes, we have eliminated a small number of errors in the first edition. Finally, we would like to thank Ying Chen, Ekaterina Ignatieva and Kai Detlefsen for the text management.

Berlin, Kaiserslautern and Louvain-la-Neuve, August 2007

# Preface to the First Edition

Until about the 1970s, financial mathematics has been rather modest compared with other mathematical disciplines. This changed rapidly after the path-breaking works of F. Black, M. Scholes, and R. Merton on derivative pricing, for which they received the Nobel prize of economics in 1997. Since 1973, the publication year of the famous Black and Scholes article, the importance of derivative instruments in financial markets has not ceased to grow. Higher risks associated with, for example, flexible instead of fixed exchange rates after the fall of the Bretton Woods system required a risk management and the use of hedging instruments for internationally active companies. More recently, globalization and the increasingly complex dependence of financial markets are reasons for using sophisticated mathematical and statistical methods and models to evaluate risks.

The necessity to improve and develop the mathematical foundation of existing risk management was emphasized in the turbulent 1990s with, for example, the Asian crisis, the hedging disasters of Metallgesellschaft and Orange County, and the fall of the Long-Term Capital Management hedge fund (controlled by Merton and Scholes!). This saw the legislator obliged to take action. In continental Europe, this development is mainly influenced by the Basel Committee on Banking Supervision, whose recommendations form the basis in the European Union for legislation, with which financial institutions are obliged to do a global, thorough risk management. As a result, there is an increasing demand for experts in financial engineering, who control risks internally, search for profitable investment opportunities and guarantee the obligations of legislation. In the future, such risk management is likely to become obligatory for other, deregulated markets such as telecommunication and energy markets. Being aware of the increasing price, volume, and credit risks in these markets, large companies usually have already created new departments dealing with asset and liability management as well as risk management.

The present text is supposed to deliver the necessary mathematical and statistical basis for a position in financial engineering. Our goal is to give a comprehensive introduction into important ideas of financial mathematics and statistics. We do not aim at covering all practically relevant details, and

we also do not discuss the technical subtleties of stochastic analysis. For both purposes there is already a vast variety of textbooks. Instead, we want to give students of mathematics, statistics, and economics a primer for the modelling and statistical analysis of financial data. Also, the book is meant for practitioners, who want to deepen their acquired practical knowledge. Apart from an introduction to the theory of pricing derivatives, we emphasize the statistical aspects of mathematical methods, i.e., the selection of appropriate models as well as fitting and validation using data.

The present book consists of three parts. The first two are organized such that they can be read independently. Each one can be used for a course of roughly 30 hours. We deliberately accept an occasional redundancy if a topic is covered in both parts but from a different perspective. The third part presents selected applications to current practical problems. Both *option pricing* as *statistical modelling of financial time series* have often been topic of seminars and lectures in the international study program *financial mathematics* of Universität Kaiserslautern (http://www.mathematik.uni-kl.de) as well as in the economics and statistics program of Humboldt-Universität zu Berlin (http://ise.wiwi.hu-berlin.de). Moreover, they formed the basis of lectures for banking practitioners which were given by the authors in various European countries.

The first part covers the classical theory of pricing derivatives. Next to the Black and Scholes option pricing formula for conventional European and American options and their numerical solution via the approximation using binomial processes, we also discuss the evaluation of some exotic options. Stochastic models for interest rates and the pricing of interest rate derivatives conclude the first part. The necessary tools of stochastic analysis, in particular the Wiener process, stochastic differential equations and Itô's Lemma will be motivated heuristically and not derived in a rigorous way. In order to render the text accessible to non-mathematicians, we do not explicitly cover advanced methods of financial mathematics such as martingale theory and the resulting elegant characterization of absence of arbitrage in complete markets.

The second part presents the already classical analysis of financial time series, which originated in the work of T. Bollerslev, R. Engle, and C. Granger. Starting with conventional linear processes, we motivate why financial time series rarely can be described using such linear models. Alternatively, we discuss the related model class of stochastic volatility models. Apart from standard ARCH and GARCH models, we discuss extensions that allow for an asymmetric impact of lagged returns on volatility. We also review multivariate GARCH models that can be applied, for example, to estimate and test

the capital asset pricing model (CAPM) or to portfolio selection problems. As a support for explorative data analysis and the search and validation of parsimonious parametric models, we emphasize the use of nonparametric models for financial time series and their fit to data using kernel estimators or other smoothing methods.

In the third part of the book, we discuss applications and practical issues such as option pricing, risk management, and credit scoring. We apply flexible GARCH type models to evaluate options and to overcome the Black and Scholes restriction of constant volatility. We give an overview of Value at Risk (VaR) and backtesting, and show that copulas can improve the estimation of VaR. A correct understanding of the statistical behavior of extremes such as September 11, 2001, is essential for risk management, and we give an overview of extreme value theory with financial applications. As a particularly popular nonparametric modelling tool in financial institutions, we discuss neural networks from a statistical viewpoint with applications to the prediction of financial time series. Next, we show how a principal components analysis can be used to explain the dynamics of implied volatilities. Finally, we present nonparametric extensions of conventional discrete choice models and apply them to the credit scoring problem.

We decided to collect some technical results concerning stochastic integration in the appendix. Here we also present Girsanov's theorem and the martingale representation theorem, with which dynamic portfolio strategies as well as an alternative proof of the Black and Scholes formula are developed. This appendix is based on work by Klaus Schindler, Saarbrücken.

In designing the book as e-book, we are going new ways of scientific publishing together with Springer Verlag and MD*Tech. The book is provided with an individual license key, which enables the reader to download the html and pdf versions of the text as well as all slides for a 60 to 90 hours lecture from the e-book server at http://www.quantlet.com. All examples, tables and graphs can be reproduced and changed interactively using the XploRe quantlet technology.

The present book would not exist without the cooperating contributions of P. Čížek, M. Fengler, Z. Hlávka, E. Kreutzberger, S. Klinke, D. Mercurio and D. Peithmann. The first part of the book arose from an extended vocational training which was developed together with G. Maercker, K. Schindler and N. Siedow. In particular, we want to thank Torsten Kleinow, who accompanied the writing of the text in all phases, developed the e-book platform and improved the presentation by various valuable contributions. Important impulses for an improved presentation were given by Klaus Schindler of the University of Saarbrücken, which we gratefully acknowledge. The chapter

on copulae is based on a contribution by Jörn Rank, Andersen Consulting, and Thomas Siegl, BHF Bank, which we adopted with their kind approval. The quantlets for multivariate GARCH models were contributed by Matthias Fengler and Helmut Herwartz. All graphs were created by Ying Chen, who also led the text management. We would like to express our thanks to these colleagues. We also benefitted from many constructive comments by our students of the universities in Kaiserslautern, Berlin, and Rotterdam. As an example of their enthusiasm we depict the preparation sheet of a student for the exam at the front pages of the book. Graphs and formulae are combined to create a spirit of the "art of quantitative finance".

Finally, for the technical realization of the text we want to thank Beate Siegler and Anja Ossetrova.

Kaiserslautern, Berlin and Rotterdam, April 2004

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# Part I

# Option Pricing

# 1 Derivatives

Classic financial mathematics deals first and foremost with basic financial instruments such as stocks, foreign currencies and bonds. A *derivative* (*derivative security* or *contingent claim*) is a financial instrument whose value depends on the value of other, more basic *underlying* variables. In this chapter we consider forward contracts, futures contracts and options as well as some combinations.

Simple derivatives have been known on European stock exchanges since the turn of the 19th century. While they lost popularity between World War I and II, they revived in the seventies with the help of work by Black, Scholes and Merton, who developed a theoretical foundation to price such instruments. Their entrepreneurial approach–which is not only applied to price derivatives but everywhere in finance where the risk of complex financial instruments is measured and controlled, received a Nobel price for economics in 1997. At the same time, it triggered the development of modern financial mathematics the basics of which is described in chapter 6 of this book. Since we have only concentrated on the mathematical modelling ideas, relevant financial terminology is only introduced when required. Numerous details which are of practical importance but which are of no interest for mathematical modelling have been left out; and refer to, for example, Hull (2000), Welcker, Kloy and Schindler (1992).

Particularly simple derivative securities are *forward* and *future contracts*. Both contracts are agreements involving two parties and call for future delivery of an asset at an agreed price. Stocks, currencies and bonds, as well as agricultural products (grain, meat) and raw materials (oil, copper, electric energy) are underlying in the contract.

**Definition 1.1 (Forward contract)**
*A* forward contract *is an agreement between two parties in which one of the parties assumes a long-term position (the other party assumes a short-term position) and obliges to purchase (sell) the underlying asset at a specified future date $T > t$, (*expiration date *or* maturity*) for a specified price $K$ (*delivery price*).*

*At time $t$, the value $V_{K,T}(S_t, \tau)$ of such a contract depends on the current value of the underlying $S_t$, the time to maturity $\tau = T - t$ and of the parameters $K$, $T$ specified in the contract.*

Futures contracts closely resemble forward contracts. While the latter do not entail any further payments until maturity, once the agreement is signed, futures contracts are traded on an exchange and mark to the market on a daily basis. Under certain circumstances forward and futures prices are identical.

**Example 1.1**
*An investor enter into a long-term forward contract on September 1, 2003, which obliges him to buy 1 000 000* EUR *at a specified exchange rate of 1.2* USD/EUR *in 90 days. The investor gains if the exchange rate is up to 1.3* USD/EUR *on November 30, 2003, since he can sell the 1 000 000* EUR *for* USD *1 300 000.*
*In this case $t$ = September 1, 2003 $\tau$ = 90 days, $T$ = November 30, and $K$ =* USD *1 200 000.*

**Definition 1.2 (Spot Price, Forward Price, Future Price)**
*The current price of the underlying (stock, currency, raw material) $S_t$ is often referred to as the* spot price*. The delivery price giving a forward contract, a value of zero is called the* forward price *and denoted $F_t$. That is, $F_t$ solves $V_{F_t,T}(S_t, \tau) = 0$. The* future price *is defined accordingly.*

Later we will compute the value of a forward contract, which determines the forward price. Since under certain circumstances forward and future contracts have the same value, their prices are equal. When such a contract is initiated in time $t = 0$, often the delivery price is set to $K = F_0$. The contract has a value of zero for both the seller and the buyer, i.e. no payments occur. Over time, as additional transactions take place on the exchange, the delivery price $K$ and the forward price $F_t$ can be different.

Contrary to forward and futures contracts where both parties are obliged to carry out the transaction, an option gives one party the right to buy or sell the security. Obviously, it's important to distinguish whether the buyer or seller of the option has the right to transact. There are two types of options: call options and put options. Furthermore, European options are delimited from American options. While European options are like forward contracts, American options can be exercised at any date before maturity. These terms are derived from historical, not geographical roots.

**Definition 1.3 (Call Option, Put Option)**
*A* European call option *is an agreement which gives the holder the right to buy the underlying asset at a specified date $T > t$, (*expiry date *or* maturity*), for a specified price $K$, (*strike price *or* exercise price*). If the holder does not exercise, the option expires as worthless.*

*A* European put option *is an agreement which gives the holder the right to sell the underlying asset at a specified date $T$ for a specified price $K$.*

*The holder of an* American call *or* put option *has the right to exercise the option at any time between $t$ and $T$.*

The option types defined above are also called *plain vanilla options*. In practice, many more complex derivatives exist and numerous new financial instruments are still emerging. *Over-the-counter (OTC) derivatives* are tailor made instruments designed by banking institutions to satisfy a particular consumer need. A compound option, for example, is such an OTC–derivative. It gives the holder the right to buy or sell at time $T$ an underlying option which matures in $T' > T$. The mathematical treatment of these *exotic options* is particularly difficult, since the current value of this instrument does not only depend on the value of the underlying $S_t$ but also on the entire path of the underlying, $S_{t'}, 0 \leq t' \leq t$.

Asian, lookback and knock–out options are path-dependent derivatives. While the delivery price $K$ of an asian option depends on the average value of the security of a certain period of time, it depends, in the case of a lookback option, on the minimum or maximum value of the security for a certain period of time. Knock–out options expire as worthless if the price level ever reaches a specified level.

To get used to forward and futures contracts, plain vanilla options and simple combinations of them, it is convenient to have a look at the *payoff* of an instrument, i.e. the value of the derivative at maturity $T$. The payoff of a long position in a forward contract is just $S_T - K$, with $S_T$ the security's spot price at expiration date $T$. The holder of the contract pays $K$ for the security and can sell it for $S_T$. Thus, he makes a profit if the value of the security $S_T$ at expiration is greater than the delivery price $K$. Being short in a forward contract implies a payoff $K - S_T$. Both payoff functions are depicted in Figure 1.1.

The call option payoff function is denoted:

$$\max\{S_T - K, 0\} = (S_T - K)^+.$$

Thus, the option holder only exercises if the delivery price $K$ is less than the value of the security $S_T$ at the expiry date $T$. In this case, he would

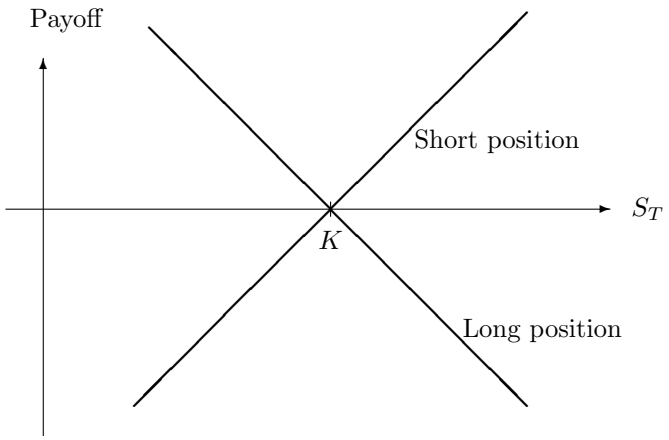



Figure 1.1: Value of forward contract at maturity

receive the same cash amount as in the case of a forward or future contract. If $K > S_T$, he will clearly choose not to exercise and the option expires as worthless. The put option payoff function is:

$$\max\{K - S_T, 0\} = (K - S_T)^+.$$

In contrast to forward and future contracts, options need to be bought for a positive amount $C(S_0, T)$, called the *option price* or *option prime*. Often, the options profit function is defined as $(S_T - K)^+ - C(S_0, T)$. However, this definition adds cash flows of different points in time. The correct profit is obtained by compounding the cash outflow in time $t = 0$ up to time $t = T$, since the investor could have invested the option at the risk–free interest rate $r$. Assuming continuous compounding at a constant interest rate $r$, the profit function of a call option is denoted: $(S_T - K)^+ - C(S_0, T)e^{rT}$.

**Example 1.2**
*Consider a short call option with delivery price $K$ and option price $C_0$ in time $t = 0$. The payoff and profit function are given in Figures 1.2 and 1.3, respectively.*

**Example 1.3**
*Combining a long call and a long put with the same delivery price, $K$ is called*

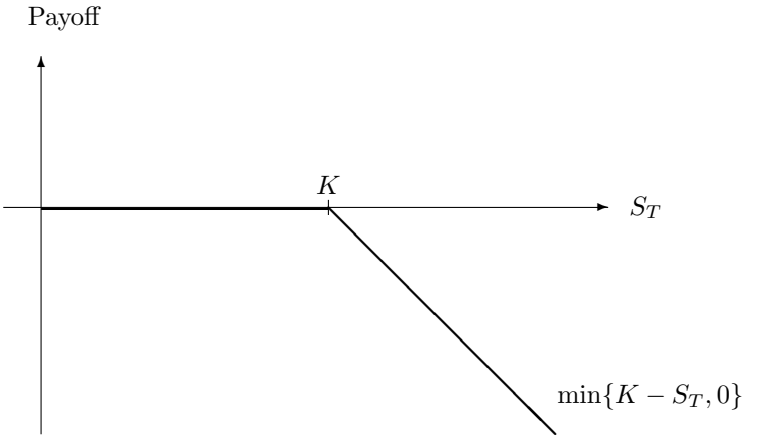



Figure 1.2: Payoff of a short position in a call option

*a* straddle. *Figure 1.4 shows the straddle profit function. $C_0$ and $P_0$ denote the call and put option option respectively.*

Another fundamental financial instrument which is used in option pricing is a *bond*. Apart from interest yields, the bond holder could receive coupon payments at fixed points in time. In particular, we will consider zero-coupon bonds, i.e. bonds which promise a single payment at a fixed future date.

**Definition 1.4 (Zero coupon Bond, Discount Bond)**
*A* zero coupon bond *or* discount bond *is a bond without coupon payments which pays interest $r$. The investor pays in time 0 an amount $B_0$ and receives at maturity $T$ the amount $B_T$ which is the sum of $B_0$ and the interest earned on $B_0$. The bonds' value at maturity is termed* face value.

Buying a zero–coupon bond corresponds to lending money at a fixed interest rate for a fixed period of time. Conversely, selling a zero-coupon bond is equivalent to borrowing money at rate $r$. Since bonds are traded on an exchange, they can be sold prior to maturity at price $B_t$, i.e. $B_0$ plus accrued interest up to time $t$.

In practice, interest rates are compounded at discrete points in time, for

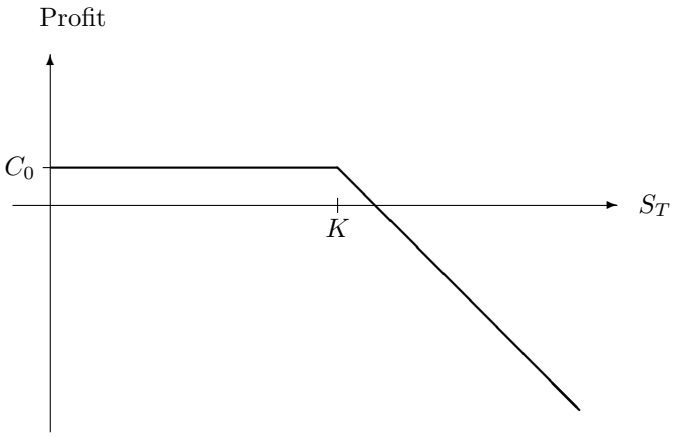



Figure 1.3: Profit of a short position in a call option

example annually, semiannually or monthly. If the interest rate $r$ is compounded annually, the initial investment $B_0$ has ($n$ years later) a value of $B_n^{(1)} = B_0(1+r)^n$. If it is compounded $k$ times per annum (p.a.), the investment pays an interest rate of $\frac{r}{k}$ each $\frac{1}{k}$ years, and has a terminal value of $B_n^{(k)} = B_0(1+\frac{r}{k})^{nk}$ after $n$ years. However, when options and other complex derivatives are priced, continuous compounding is used, which denoted for $k \to \infty$. In this case, the initial investment $B_0$ grows in $n$ years to $B_n = B_0 \cdot e^{nr}$, and $r$ is called *short rate*. The difference between discrete and continuous compounding is small when $k$ is large. While an investment of $B_0 = 1000$ EUR at a yearly rate $r = 10\%$ grows to 1100 EUR within a year when annually compounded, it grows to 1105.17 EUR when continuously compounded.

In light of this, the continuous compounded rate $r$ can be modified to account for these deviations. Assuming annual compounding at rate $r_1$, for both continuous and annual compounding, a continuous compounded rate $r = \log(1+r_1)$ has to be applied, in order to obtain the same terminal value $B_n = B_n^{(1)}$.

If not stated otherwise, continuous compounding will be assumed from here on. For comparing cash flows occurring at different points in time, they

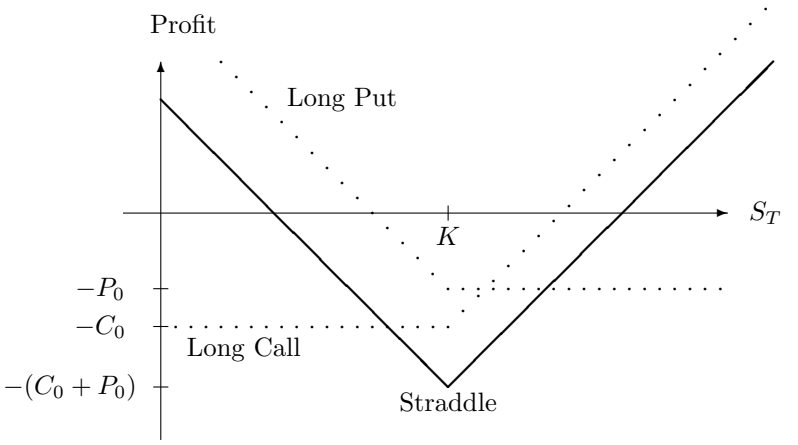



Figure 1.4: Profit of a straddle

need to be compounded or discounted to the same point in time. That is, interest payments are added or subtracted. With continuous compounding, an investment of $B$ in time $t$ in $\Delta t > 0$ is

**compounded** to time $t + \Delta t$: $B\, e^{r\Delta t}$

**discounted** to time $t - \Delta t$: $B\, e^{-r\Delta t}$.

Some more financial terms will be introduced before the end of the chapter. A *portfolio* is a combination of one or more financial instruments - its value is considered as an individual financial instrument. One element of a portfolio is also called a *position*. An investor assumes a *long position* when he buys an instrument, and a *short position* when he sells it. A *long call* results from buying a call option, a *long put* from buying a put option, and *a short forward* from selling a forward contract.

An investor closes out a position of his portfolio by making the future portfolio performance independent of the instrument. If the latter is traded on an exchange, he can sell (e.g. a stock or a bond) or buy (e.g. borrowed money) it. However, if the instrument is traded, the investor can close out the position by adding to the portfolio the inverse instrument. Thus, both add up to zero, and do not influence the portfolio performance any further.

**Example 1.4**
*Consider an investor who on 1 February bought a 1 000 000* USD *forward contract with a delivery price of 1 200 000* EUR *with maturity within one year. On 1 June, he wishes to close out the position. He can* sell *another forward contract of the same size with the same delivery price and the maturity date, namely 31 January. The long and the short positions add up to zero at any point in time.*

*Short selling* is a trading strategy that involves selling financial instruments, for example stocks, which an investor does not own. At a later stage, the investor buys back these objects. In practice, this requires the intervention of a broker who mediates with another client who owes the objects and is willing to lend them to the investor. The short selling investor commits to pay the client any foregone income, (dividends for example), that is earned in the interim.

**Example 1.5**
*An investor selling short 1000 stocks, lends them from the owner and sells them immediately for 1000 $S_0$ in the market ($S_t$ denotes the stock price at time $t$). Later, at time $t > 0$, he closes out the position, by buying back the stocks for 1000 $S_t$ and returns them to the owner. The strategy is profitable if $S_t$ is clearly below $S_0$. If in time $t_0$, $0 < t_0 < t$, a dividend $D$ per share is paid, the investor pays 1000 $D$ to the owner. Short selling is in practice subject to numerous restrictions. In this example, it is only the possibility of short selling that will be of any interest.*

## 1.1 Recommended Literature

Basic textbooks on derivatives are, among others, Hull (2000), Jarrow (1992) and Cox and Rubinstein (1985). Neftci (1996) and Duffie (1996) are more advanced on a mathematical level. A rather practical but still theoretically well–founded introduction, is provided by Briys, Bellalah, Mai and de Varenne (1998).

# 2 Introduction to Option Management

## 2.1 Arbitrage Relations

In this section we consider the fundamental notion of no–arbitrage. An *arbitrage opportunity* arises if it is possible to make a riskless profit. In an ideal financial market, in which all investors dispose of the same pieces of information and in which all investors can react instantaneously, there should not be any arbitrage opportunity. Since otherwise each investor would try to realize the riskless profit instantaneously. The resulting transactions would change the prices of the involved financial instruments such that the arbitrage opportunity disappears.

Additionally to no–arbitrage we presume in the remaining chapter that the financial market fulfills further simplifying assumptions which are in this context of minor importance and solely serve to ease the argumentation. If these assumptions hold we speak of a perfect financial market.

**ASSUMPTION** *(perfect financial market)*
*There are no arbitrage opportunities, no transaction costs, no taxes, and no restrictions on short selling. Lending rates equal borrowing rates and all securities are perfectly divisible.*

The assumption of a perfect financial market is sufficient to determine the value of future and forward contracts as well as some important relations between the prices of some types of options. Above all no mathematical model for the price of the financial instrument is needed. However, in order to determine the value of options more than only economic assumptions are necessary. A detailed mathematical modelling becomes inevitable. Each mathematical approach though has to be in line with certain fundamental arbitrage relations being developed in this chapter. If the model implies values of future and forward contracts or option prices which do not fulfill these relations the model's assumptions must be wrong.

An important conclusion drawn from the assumption of a perfect financial market and thus from no–arbitrage will be used frequently in the proofs to come. It is the fact that two portfolios which have at a certain time $T$ the same value must have the same value at a prior time $t < T$ as well. Due to its importance we will further illustrate this reasoning. We proceed from two portfolios $A$ and $B$ consisting of arbitrary financial instruments. Their value in time $t$ will be denoted by $W_A(t)$ and $W_B(t)$ respectively. For any fixed point of time $T$, we assume that $W_A(T) = W_B(T)$ independently of the prior time $T$ values of each financial instrument contained in $A$ and $B$. For any prior point of time $t < T$ we assume without loss of generality that $W_A(t) \leq W_B(t)$. In time $t$ an investor can construct without own financial resources a portfolio which is a combination of $A$ and $B$ by buying one unit of every instrument of $A$, selling one unit of every instrument of $B$ (short selling) and by investing the difference $\Delta(t) = W_B(t) - W_A(t) \geq 0$ at a fixed rate $r$. The combined portfolio has at time $t$ a value of

$$W_A(t) - W_B(t) + \Delta(t) = 0,$$

i.e. the investor has no initial costs. At time $T$ the part of the combined portfolio which is invested at rate $r$ has the compounded value $\Delta(T) = \Delta(t)e^{r(T-t)}$, and hence the combined portfolio has a value of

$$W_A(T) - W_B(T) + \Delta(T) = \Delta(t)e^{r(T-t)} > 0,$$

if $\Delta(t) > 0$. The investor made a riskless gain by investing in the combined portfolio which contradicts the no–arbitrage assumption. Therefore, it must hold $\Delta(t) = 0$, i.e. $W_A(t) = W_B(t)$.

The previous reasoning can be used to determine the unknown value of a financial derivative. For this, a portfolio $A$ is constructed which contains instruments with known price along with one unit of the derivative under investigation. Portfolio $A$ will be compared to another portfolio $B$, called the *duplicating portfolio*, which contains exclusively instruments with known prices. Since the duplicating portfolio $B$ is constructed such that for certain it has the same value at a fixed point of time $T$ as portfolio $A$ the no–arbitrage assumption implies that both portfolios must have the same value at any prior point of time. The value of the financial derivative can thus be computed at any time $t \leq T$. We illustrate this procedure in the following example of a forward contract.

**Theorem 2.1**
*We consider a long forward contract to buy an object which has a price of $S_t$ at time $t$. Let $K$ be the delivery price, and let $T$ be the maturity date. $V(s, \tau)$*

*denotes the value of the long forward contract at time $t$ as a function of the current price $S_t = s$ and the time to maturity $\tau = T - t$. We assume constant interest rates $r$ during the time to maturity.*

1. *If the underlying object does not pay any dividends and does not involve any costs during the time to maturity $\tau$, then it holds*

$$V(S_t, \tau) = V_{K,T}(S_t, \tau) = S_t - Ke^{-r\tau} \tag{2.1}$$

   *The forward price is equal to $F_t = S_t e^{r\tau}$.*

2. *If during the time to maturity the underlying pays at discrete time points dividends or involves any costs whose current time $t$ discounted total value is equal to $D_t$, then it holds*

$$V(S_t, \tau) = V_{K,T}(S_t, \tau) = S_t - D_t - Ke^{-r\tau} \tag{2.2}$$

   *The forward price is equal to $F_t = (S_t - D_t)e^{r\tau}$.*

3. *If the underlying involves continuous costs at rate $b$, then it holds*

$$V(S_t, \tau) = V_{K,T}(S_t, \tau) = S_t e^{(b-r)\tau} - Ke^{-r\tau} \tag{2.3}$$

   *The forward price is equal to $F_t = S_t e^{b\tau}$.*

**Proof:**
For simplicity we assume the underlying object to be a stock paying either discrete dividend yields whose value discounted to time $t$ is $D_t$ or paying a continuous dividend yield at rate $b$. In the latter case the stock involves continuous costs equal to $b = r - d$. The investor having a long position in the stock gains dividends (as negative costs) at rate $d$ but simultaneously loses interests at rate $r$ since he invested his capital in the stock instead of in a bond with a fixed interest rate. In place of stocks, bonds, currencies or other simple instruments can be considered as well.

1. We consider at time $t$ the following two portfolios $A$ and $B$:

**Portfolio** $A$**:** One long forward contract on a stock with delivery price $K$, maturing in time $T$.
One long zero bond with face value $K$, maturing in time $T$.

**Portfolio** $B$**:** A long position in one unit of the stock.

At maturity $T$ portfolio $A$ contains a zero bond of value $K$. Selling this zero bond for $K$ the obligation to buy the stock for $K$ can be fulfilled. Following these transactions portfolio $A$ consists as well as portfolio $B$ of one unit of the stock. Thus both portfolios have at time $T$ the same value and must

therefore, due to the no–arbitrage assumption, have the same value at any time $t$ prior to $T$ :

$$V(S_t, \tau) + Ke^{-r\tau} = S_t \ , \tag{2.4}$$

since the value of the zero bond at time $t$ is given by discounting $K$ at rate $r$, $Ke^{-r\tau}$. The forward price is by definition the solution of

$$0 = V_{F_t,T}(S_t, \tau) = S_t - F_t e^{-r\tau}.$$

2. We consider at time $t$ the two portfolios $A$ and $B$ as given above and add one position to portfolio $B$ :

**Portfolio** $B$**:** A long position in one unit of the stock and one short position of size $D_t$ in a zero bond with interest rate $r$ (lending an amount of money of $D_t$).

At maturity $T$ the dividend yields of the stock in portfolio $B$, which compounded to time $T$ amount to $D_t e^{r\tau}$, are used to pay back the bond. Thus, both portfolios $A$ and $B$ consist again of one unit of the stock, and therefore they must have the same value at any time $t < T$ :

$$V(S_t, \tau) + Ke^{-r\tau} = S_t - D_t \ . \tag{2.5}$$

The forward price results as in part 1 from the definition.

3. If the stock pays dividends continuously at a rate $d$, then the reasoning is similar as in part 2. Once again, we consider at time $t$ two portfolios $A$ and $B$. And again, $A$ is left unchanged, $B$ is now composed of the following position:

**Portfolio** $B$**:** A long position in $e^{-d\tau}$ stocks.

Reinvesting the dividends yields continuously in the stock portfolio $B$ consists again of exactly one stock at time $T$. Heuristically, this can be illustrated as follows: In the time interval $[t, t+\delta]$ the stock pays approximately, for a small $\delta$, a dividend of $d \cdot \delta \cdot S_t$. Thus, the current total amount of stocks in the portfolio, $e^{-d\tau} = e^{-d(T-t)}$, pays a total dividend yield of $d \cdot \delta \cdot S_t \cdot e^{-d(T-t)}$, which is reinvested in the stock. Assuming that the stock price does not change significantly in the interval $[t, t+\delta]$, i.e. $S_{t+\delta} \approx S_t$, portfolio $B$ contains in time $t + \delta$

$$(1 + d \cdot \delta) \cdot e^{-d(T-t)} \approx e^{d\delta} \cdot e^{-d(T-t)} = e^{-d(T-t-\delta)}$$

stocks. The above reasoning can be done exactly by taking the limit $\delta \to 0$, and it can be shown that portfolio $B$ contains at any time $s$ between $t$ and $T$ exactly $e^{-d(T-s)}$ stocks. That is, for $s = T$ portfolio $B$ is composed of

exactly one stock. The same reasoning as in part 1 leads to the conclusion that portfolio $A$ and $B$ must have the same value at any time $t$. Thus, we have

$$V(S_t, \tau) + Ke^{-r\tau} = e^{-d\tau} S_t \ . \tag{2.6}$$

where we have to set $b = r - d$. The forward price results as in part 1 from the definition. □

**Example 2.1** *We consider a long forward contract on a 5 year bond which is currently traded at a price of 900* EUR*. The delivery price is 910* EUR*, the time to maturity of the forward contract is one year. The coupon payments of the bond of 60* EUR *occur after 6 and 12 months (the latter shortly before maturity of the forward contract). The continuously compounded annual interest rates for 6 and 12 months are 9% and 10% respectively. In this example we have*

$$S_t = 900 \ , \ K = 910 \ , \ r = 0.10 \ , \ \tau = 1 \ , \ D_t = 60e^{-0.09 \cdot \frac{1}{2}} + 60e^{-0.10} = 111.65 \tag{2.7}$$

*Thus, the value of the forward contract is given by*

$$V(S_t, \tau) = 900 - 111.65 - 910e^{-0.10} = -35.05. \tag{2.8}$$

*The value of the respective short position in the forward contract is +35.05. The price $F_t$ of the forward contract is equal to $F_t = (S_t - D_t)e^{r\tau} = 871.26$.*

**Example 2.2** *Consider a long forward contract to buy 1000 Dollar. If the investor buys the 1000 Dollar and invests this amount in an American bond, the American interest rate can be interpreted as a dividend yield $d$ which is continuously paid. Let $r$ be the home interest rate. The investment involves costs $b = r - d$, which are the difference between the American and the home interest rate. Denoting the dollar exchange rate by $S_t$ the price of the forward contract is then given by*

$$F_t = S_t e^{b\tau} = S_t e^{(r-d)\tau}. \tag{2.9}$$

*While for $r > d$ a report $S_t < F_t$ results, for $r < d$ a backwardation $S_t > F_t$ results. If $r > d$ and the delivery price is chosen to equal the current exchange rate, i.e. $K = S_t$, then the value of the forward contract is*

$$V_{S_t,T}(S_t, \tau) = S_t(e^{-d\tau} - e^{-r\tau}) > 0.$$

*Buying the forward contract at a price of $S_t$ is thus more expensive than buying the dollars immediately for the same price since in the former case the investor can invest the money up to time $T$ in a domestic bond paying an interest rate which is higher than the American interest rate.*

The following result states that forward and future contracts with the same delivery price and the same time to maturity are equal, if interest rates are constant during the contract period. We will use the fact that by definition forward and future contracts do not cost anything if the delivery price is chosen to be equal to the current price of the forward contract respectively the price of the future contract.

**Theorem 2.2**
*If interest rates are constant during contract period, then forward and future prices are equal.*

**Proof:**
We proceed from the assumption that the future contract is agreed on at time 0, and that is has a time to maturity of $N$ days. We assume that profits and losses are settled (marked to market) on a daily basis at a daily interest rate of $\rho$. While the forward price at the end of day 0 is denoted by $F$, the future price at the end of day $t$, $t = 0, 1, \cdots, N$ is denoted by $F_t$. The goal is to show that $F = F_0$. For that we construct two portfolios again:

**Portfolio** $A$**:** A long position in $e^{N\rho}$ forward contracts with delivery price $F$ and maturity date $N$.
A long position in a zero bond with face value $Fe^{N\rho}$ maturing in $N$ days.

**Portfolio** $B$**:** A long position in futures contracts with delivery price $F_t$ and maturity date $N$. The contracts are bought daily such that the portfolio contains at the end of the $t$–th day exactly $e^{(t+1)\rho}$ future contracts $(t = 0, 1, \cdots, N)$.
A long position in a zero bond with face value $F_0 e^{N\rho}$ maturing in $N$ days.

Purchasing a forward or a future contract does not cost anything since their delivery prices are set to equal the current forward or future price. Due to the marking to market procedure the holder of portfolio $B$ receives from day $t-1$ to day $t$ for each future contract an amount of $F_t - F_{t-1}$ which can possibly be negative (i.e. he has to pay).

At maturity, i.e. at the end of day $N$, the zero bond of portfolio $A$ is sold at the face value $Fe^{N\rho}$ to fulfill the terms of the forward contract and to buy $e^{N\rho}$ stocks at a the delivery price $F$. Then $A$ contains exclusively these stocks and has a value of $S_N e^{N\rho}$. Following, we show that portfolio $B$ has the same value.

At the beginning of day $t$ portfolio $B$ contains $e^{t\rho}$ future contracts, and the holder receives due to the marking to market procedure the amount

$(F_t - F_{t-1})e^{t\rho}$ which can possibly be negative. During the day he increases his long position in the future contracts at zero costs such that the portfolio contains $e^{(t+1)\rho}$ future contracts at the end of the day. The earnings at day $t$ compounded to the maturity date have a value of:

$$(F_t - F_{t-1})e^{t\rho} \cdot e^{(N-t)\rho} = (F_t - F_{t-1})e^{N\rho}. \tag{2.10}$$

At maturity the terms of the future contracts are fulfilled due to the marking to market procedure. All profits and losses compounded to day $N$ have a value of:

$$\sum_{t=1}^{N}(F_t - F_{t-1})e^{N\rho} = (F_N - F_0)e^{N\rho}. \tag{2.11}$$

Together with the zero bond portfolio $B$ has at day $N$ a value of

$$(F_N - F_0)e^{N\rho} + F_0 e^{N\rho} = F_N e^{N\rho} = S_N e^{N\rho},$$

since at maturity the future price $F_N$ and the price $S_N$ of the underlying are obviously equal.

Hence, both portfolios have at day $N$ the same value and thus due to the no–arbitrage assumption their day 0 values must be equal as well. Since the forward contract with delivery price $F$ has a value of 0 at day 0 due to the definition of the forward price, the value of portfolio $A$ is equal to the value of the zero bond, i.e. F (face value $Fe^{N\rho}$ discounted to day 0). Correspondingly, the $e^{\rho}$ futures contained in portfolio $B$ have at the end of day 0 a value of 0 due to the definition of the future price. Again, the value of portfolio $B$ reduces to the value of the zero bond. The latter has a value of $F_0$ (face value $F_0e^{N\rho}$ discounted to day 0). Putting things together, we conclude that $F = F_0$. □

Now, we want to proof some relationship between option prices using similar methods. The most elementary properties are summarized in the following remark without a proof. For that, we need the notion of the *intrinsic value* of an option.

**Definition 2.1 (Intrinsic Value)**
*The* intrinsic value *of a call option at time $t$ is given by* $\max(S_t - K, 0)$*, the intrinsic value of a put option is given by* $\max(K - S_t, 0)$*. If the intrinsic value of an option is positive we say that the option is* in the money*. If $S_t = K$, then the option is* at the money*. If the intrinsic value is negative, then the option is said to be* out of the money*.*

**Remark 2.1**
*Options satisfy the following elementary relations.* $C(s,\tau) = C_{K,T}(s,\tau)$ *and* $P(s,\tau) = P_{K,T}(s,\tau)$ *denote the time* $t$ *value of a call and a put with delivery price* $K$ *and maturity date* $T$*, if* $\tau = T - t$ *is the time to maturity and the price of the underlying is* $s$*, i.e.* $S_t = s$*.*

1. *Option prices are non negative since an exercise only takes place if it is in the interest of the holder. An option gives the right to exercise. The holder is not obligated to do so.*
2. *American and European options have the same value at maturity* $T$ *since in* $T$ *they give the same rights to the holder. At maturity* $T$ *the value of the option is equal to the intrinsic value:*

$$C_{K,T}(S_T, 0) = \max(S_T - K, 0)\ , \quad P_{K,T}(S_T, 0) = \max(K - S_T, 0).$$

3. *An American option must be traded at least at its intrinsic value since otherwise a riskless profit can be realized by buying and immediately exercising the option. This relation does not hold in general for European options. The reason is that a European option can be exercised only indirectly by means of a future contract. The thereby involved discounting rate can possibly lead to the option being worth less than its intrinsic value.*
4. *The value of two American options which have different time to maturities,* $T_1 \le T_2$*, is monotonous in time to maturity:*

$$C_{K,T_1}(s, T_1 - t) \le C_{K,T_2}(s, T_2 - t)\ , \quad P_{K,T_1}(s, T_1 - t) \le P_{K,T_2}(s, T_2 - t).$$

*This follows, for calls, say, using 2., 3. from the inequality which holds at time* $t = T_1$ *with* $s = S_{T_1}$

$$C_{K,T_2}(s, T_2 - T_1) \ge \ \textit{intrinsic value}\ = \max(s - K, 0) = C_{K,T_1}(s, 0) \tag{2.12}$$

*Due to the no–arbitrage assumption the inequality must hold for any point in time* $t \le T_1$*. For European options this result does not hold in general.*

5. *An American option is at least as valuable as the identically specified European option since the American option gives more rights to the holder.*
6. *The value of a call is a monotonously decreasing function of the delivery price since the right to buy is the more valuable the lower the agreed upon delivery price. Accordingly, the value of a put is a monotonously increasing function of the delivery price.*

$$C_{K_1,T}(s,\tau) \ge C_{K_2,T}(s,\tau)\ , \quad P_{K_1,T}(s,\tau) \le P_{K_2,T}(s,\tau)$$

*for $K_1 \leq K_2$. This holds for American as well as for European options.*

The value of European call and put options on the same underlying with the same time to maturity and delivery price are closely linked to each other without using a complicated mathematical model.

**Theorem 2.3 (Put–Call Parity for European Options)**
*For the value of a European call and put option which have the same maturity date $T$, the same delivery price $K$, the same underlying the following holds (where $r$ denotes the continuous interest rate):*

1. *If the underlying pays a dividend yield with a time $t$ discounted total value of $D_t$ during the time to maturity $\tau = T - t$ then it holds*

$$C(S_t, \tau) = P(S_t, \tau) + S_t - D_t - Ke^{-r\tau} \tag{2.13}$$

*SFEPutCall*

2. *If the underlying involves continuous costs of carry at rate $b$ during the time to maturity $\tau = T - t$ then it holds*

$$C(S_t, \tau) = P(S_t, \tau) + S_t e^{(b-r)\tau} - Ke^{-r\tau} \tag{2.14}$$

**Proof:**
For simplicity, we again assume the underlying to be a stock. We consider a portfolio $A$ consisting of one call which will be duplicated by a suitable portfolio $B$ containing a put among others.

1. In the case of discrete dividend yields we consider at time $t$ the following portfolio $B$ :

1. Buy the put.
2. Sell a zero bond with face value $K$ maturing $T$.
3. Buy one stock.
4. Sell a zero bond at the current price $D_t$.

The stock in portfolio $B$ pays dividends whose value discounted to time $t$ is $D_t$. At time $T$ these dividend yields are used to pay back the zero bond of position d). Hence this position has a value of zero at time $T$. Table 2.1 shows the value of portfolio $B$ at time $T$ where we distinguished the situations where the put is exercised ($K \geq S_T$) and where it is not exercised. At time $T$ portfolio $B$ has thus the same value $\max(S_T - K, 0)$ as the call. To avoid arbitrage opportunities both portfolios $A$ and $B$ must have the same value at any time $t$ prior $T$, that is it holds

$$C(S_t, \tau) = P(S_t, \tau) - Ke^{-r\tau} + S_t - D_t \tag{2.15}$$

| Position | Value at time $T$ | |
|---|---|---|
| | $K < S_T$ | $K \geq S_T$ |
| long put | 0 | $K - S_T$ |
| short bond | $-K$ | $-K$ |
| long stock | $S_T$ | $S_T$ |
| short bond ($D_t$) | 0 | 0 |
| Sum | $S_T - K$ | 0 |

Table 2.1: Value of portfolio $B$ at time $T$ (Theorem 2.3).

2. In the case of continuous dividends at rate $d$ and corresponding costs of carry $b = r - d$ we consider the same portfolio $B$ as in part 1. but this time without position d). Instead we buy $e^{-d\tau}$ stocks in position c) whose dividends are immediately reinvested in the same stock. If $d$ is negative, then the costs are financed by selling stocks. Thus, portfolio $B$ contains exactly one stock at time $T$, and we conclude as in part 1. that the value of portfolio $B$ is at time $t$ equal to the value of the call. □

The proof of the put–call parity holds only for European options. For American options it may happen that the put or call are exercised prior maturity and that both portfolios are not hold until maturity.

The following result makes it possible to check whether prices of options on the same underlying are consistent. If the convexity formulated below is violated, then arbitrage opportunities arise as we will show in the example following the proof of the next theorem.

**Theorem 2.4**
*The price of a (American or European) Option is a convex function of the delivery price.*

**Proof:**
It suffices to consider calls since the proof is analogous for puts. The put–call parity for European options is linear in the term which depends explicitly on $K$. Hence, for European options it follows immediately that puts are convex in $K$ given that calls are convex in $K$.

For $0 \leq \lambda \leq 1$ and $K_1 < K_0$ we define $K_\lambda \stackrel{\text{def}}{=} \lambda K_1 + (1 - \lambda)K_0$. We consider a portfolio $A$ which at time $t < T$ consists of one call with delivery price $K_\lambda$ and maturity date $T$. At time $t$ we duplicate this portfolio by the following portfolio $B$ :

| | Value at time $t'$ | | | |
|---|---|---|---|---|
| Position | $S_{t'} \leq K_1$ | $K_1 < S_{t'} \leq K_\lambda$ | $K_\lambda < S_{t'} \leq K_0$ | $K_0 < S_{t'}$ |
| $B$ 1. | 0 | $\lambda(S_{t'} - K_1)$ | $\lambda(S_{t'} - K_1)$ | $\lambda(S_{t'} - K_1)$ |
| $B$ 2. | 0 | 0 | 0 | $(1-\lambda)(S_{t'} - K_0)$ |
| $-A$ | 0 | 0 | $-(S_{t'} - K_\lambda)$ | $-(S_{t'} - K_\lambda)$ |
| Sum | 0 | $\lambda(S_{t'} - K_1)$ | $(1-\lambda)(K_0 - S_{t'})$ | 0 |

Table 2.2: Difference in the values of portfolios $B$ and $A$ at time $t'$ (Theorem 2.4).

| Delivery price | Option price |
|---|---|
| $K_1 = 190$ | 30.6 EUR |
| $K_\lambda = 200$ | 26.0 EUR |
| $K_0 = 220$ | 14.4 EUR |

Table 2.3: Data of Example 2.3.

1. A long position in $\lambda$ calls with delivery price $K_1$ maturing in $T$.
2. A long position in $(1-\lambda)$ calls delivery price $K_0$ maturing in $T$.

By liquidating both portfolios at an arbitrary point of time $t', t \leq t' \leq T$ we can compute the difference in the values of portfolio $A$ and $B$ which is given in Table 2.2

Since $\lambda(S_{t'} - K_1) \geq 0$ und $(1-\lambda)(K_0 - S_{t'})) \geq 0$ in the last row of Table 2.2 the difference in the values of portfolio $A$ and $B$ at time $t'$ and thus for any point of time $t < t'$ is greater than or equal to zero. Hence, denoting $\tau = T - t$ it holds

$$\lambda C_{K_1,T}(S_t, \tau) + (1-\lambda) C_{K_0,T}(S_t, \tau) - C_{K_\lambda,T}(S_t, \tau) \geq 0 \tag{2.16}$$

□

**Example 2.3**
*We consider three European call options on the MD*TECH A.G. having all the same time to maturity and delivery prices* $K_1 = 190$, $K_\lambda = 200$, $K_0 = 220$, *i.e.* $\lambda = \frac{2}{3}$. *Table 2.3 shows the data of this example. Due to the last theorem it must hold:*

$$\tfrac{2}{3} C_{K_1,T}(S_t, \tau) + \tfrac{1}{3} C_{K_0,T}(S_t, \tau) \geq C_{K_\lambda,T}(S_t, \tau) \tag{2.17}$$

| | Value at time $T$ | | | |
|---|---|---|---|---|
| Position | $S_T \leq 190$ | $190 < S_T \leq 200$ | $200 < S_T \leq 220$ | $220 < S_T$ |
| 1. | 0 | $\frac{2}{3}(S_T - 190)$ | $\frac{2}{3}(S_T - 190)$ | $\frac{2}{3}(S_T - 190)$ |
| 2. | 0 | 0 | 0 | $\frac{1}{3}(S_T - 220)$ |
| 3. | 0 | 0 | $-(S_T - 200)$ | $-(S_T - 200)$ |
| Sum | 0 | $\frac{2}{3}(S_T - 190)$ | $\frac{1}{3}(220 - S_T)$ | 0 |

Table 2.4: Portfolio value at time $T$ of Example 2.3.

*Since this condition is obviously violated an arbitrage opportunity exists, and with the following portfolio a riskless gain can be realized:*

1. *A long position in* $\lambda = \frac{2}{3}$ *calls with delivery price* $K_1$.
2. *A long position in* $1 - \lambda = \frac{1}{3}$ *calls with delivery price* $K_0$.
3. *A short position in 1 call with delivery price* $K_\lambda \stackrel{\text{def}}{=} \frac{2}{3}K_1 + \frac{1}{3}K_0$.

*By setting up this portfolio at the current time t we realize an immediate profit of +0.80* EUR. *The portfolio value at options' maturity T is given by Table 2.4 from which we can extract that we realize further profits for stock prices* $S_T$ *between 190 and 220 of at most* $\frac{20}{3}$ EUR.

We already said that option prices are monotonous functions of the delivery price. The following theorem for European options is more precise on this subject.

**Theorem 2.5**
*For two European calls (puts) with the same maturity date* $T$ *and delivery prices* $K_1 \leq K_2$ *it holds at time* $t \leq T$*:*

$$0 \leq C_{K_1,T}(S_t, \tau) - C_{K_2,T}(S_t, \tau) \leq (K_2 - K_1)e^{-r\tau} \tag{2.18}$$

*or*

$$0 \leq P_{K_2,T}(S_t, \tau) - P_{K_1,T}(S_t, \tau) \leq (K_2 - K_1)e^{-r\tau} \tag{2.19}$$

*with* $\tau = T - t$ *and* $r$ *denoting the time to maturity and the interest rate respectively. If call (put) option prices are differentiable as a function of the delivery price, then by taking the limit* $K_2 - K_1 \to 0$ *it follows*

$$1 \leq -e^{-r\tau} \leq \frac{\partial C}{\partial K} \leq 0 \quad \textit{bzw.} \quad 0 \leq \frac{\partial P}{\partial K} \leq e^{-r\tau} \leq 1 \tag{2.20}$$

| | Value at time $T$ | | |
|---|---|---|---|
| Position | $S_T \leq K_1$ | $K < S_T < K_2$ | $K_2 \leq S_T$ |
| $B$ 1. | 0 | 0 | $S_T - K_2$ |
| $B$ 2. | $K_2 - K_1$ | $K_2 - K_1$ | $K_2 - K_1$ |
| $-A$ | 0 | $-(S_T - K_1)$ | $-(S_T - K_1)$ |
| Sum | $K_2 - K_1$ | $K_2 - S_T$ | 0 |

Table 2.5: Difference in the values of portfolios $B$ and $A$ at time $T$ (Theorem 2.5).

**Proof:**
We proof the theorem for calls since for puts the reasoning is analogous. For this we consider a portfolio $A$ containing one call with delivery price $K_1$ which we compare to a duplicating portfolio $B$. At time $t$ the latter portfolio consists of the following two positions:

1. A long position in one call with delivery price $K_2$.
2. A long position in one zero bond with face value $(K_2 - K_1)$ maturing in $T$.

The difference of the value of portfolios $B$ and $A$ at time $T$ is shown in Table 2.5. At time $T$ portfolio $B$ is clearly as valuable as portfolio $A$ which given the no–arbitrage assumption must hold at time $t$ as well. We conclude:

$$C_{K_2,T}(S_t, \tau) + (K_2 - K_1)e^{-r\tau} \geq C_{K_1,T}(S_t, \tau).$$

□

## 2.2 Portfolio Insurance

A major purpose of options is hedging, i.e. the protection of investments against market risk caused by random price movements. An example for active hedging with options is the portfolio insurance. That is to strike deals in order to change at a certain point of time the risk structure of a portfolio such that at a future point of time

- the positive profits are reduced by a small amount (which can be interpreted as an insurance premium) and in that way
- the portfolio value does not drop below a certain *floor*.

The portfolio insurance creates a risk structure of the portfolio which prevents extreme losses. For illustration purposes we consider at first a simple example.

**Example 2.4**
*An investor has a capital of 10 500* EUR *at his disposal to buy stocks whose current price is 100* EUR*. Furthermore, put options on the same stock with a delivery price of* $K = 100$ *and a time to maturity of one year are quoted at a market price of 5* EUR *per contract. We consider two investment alternatives.*

**Portfolio A:** *Buying 105 stocks.*

**Portfolio B:** *Buying 100 stocks for 10 000* EUR *and buying 100 put options for 500* EUR*.*

*The price of the put options can be interpreted as the premium to insure the stocks against falling below a level of 10 000* EUR*. Denoting the stock price in one year by* $S_T$ *the value of the non–insured portfolio is* $105 \cdot S_T$*. This portfolio bears the full market risk that the stock price drops significantly below 100* EUR*. The insured portfolio, however, is at least as worth as 10 000* EUR *since if* $S_T < 100$ *the holder exercises the put options and sells the 100 stocks for 100* EUR *each.*

*Should the stock price increase above 100* EUR *the investor does not exercise the put which thus expires worthless. By buying the put some of the capital of portfolio B is sacrificed to insure against high losses. But, while the probabilities of high profits slightly decrease, the probabilities of high losses decrease to zero. Investing in portfolio B the investor looses at most 500* EUR *which he paid for the put. Table 2.6 shows the impact of the stock price* $S_T$ *in one year on both the insured and the non–insured portfolio values and returns.*

The numerous conceivable strategies to insure portfolios can be classified by the frequency with which the positions in the portfolio have to rebalanced. Two approaches can be distinguished:

- Static strategies rebalance the portfolio positions at most once before expiration of the investment horizon.
- Dynamic strategies rebalance the portfolio positions very frequently, ideally continuously, according to certain rules.

The static strategy sketched in the previous example can be modified. Instead of hedging by means of put options the investor can chose between the following two strategies:

| Stock price $S_T$ [EUR] | Non–insured portfolio Value [EUR] | Return % p.a. | Insured portfolio Value [EUR] | Return % p.a. | Insured portfolio in % of the non–insured portfolio |
|---|---|---|---|---|---|
| 50 | 5250 | –50 | 10000 | –4.8 | 190 |
| 60 | 6300 | –40 | 10000 | –4.8 | 159 |
| 70 | 7350 | –30 | 10000 | –4.8 | 136 |
| 80 | 8400 | –20 | 10000 | –4.8 | 119 |
| 90 | 9450 | –10 | 10000 | –4.8 | 106 |
| 100 | 10500 | 0 | 10000 | -4.8 | 95 |
| 110 | 11550 | +10 | 11000 | +4.8 | 95 |
| 120 | 12600 | +20 | 12000 | +14.3 | 95 |
| 130 | 13650 | +30 | 13000 | +23.8 | 95 |
| 140 | 14700 | +40 | 14000 | +33.3 | 95 |

Table 2.6: The effect of portfolio insurance on portfolio value and return.

**Strategy 1:** The investor buys an equal number of *stocks and puts*.

**Strategy 2:** The investor buys *bonds* with a face value equal to the floor he is aiming at and with the remaining money buys calls on the stock.

All approaches commonly practiced rely on modifications of the above basic strategies. While following the first strategy it is the put which guarantees that the invested capital does not drop below the floor, applying the second strategy it is the bond which insures the investor against falling prices. The stocks, followed by the calls make up for the profits in the event of rising prices. The similarity of both strategies follows from the put–call parity, see Theorem 2.3.

Before deciding on what kind of portfolio insurance will be used, some points need to be clarified:

1. Which financial instruments are provided by the market, and what are their characteristics (coupons, volatilities, correlation with the market etc.)?
2. What knowledge does the investor have about:
   - the composition of the portfolio (which financial instruments),
   - the amount of capital to invest,
   - the investment horizon,
   - the floor (lower bound of the portfolio value) or rather the minimum return he is aiming at the end of the investment. Given the

| Data of Example 2.5: | |
|---|---|
| Current point of time $t$ | 0 |
| Available capital $V$ | 100 000 EUR |
| Target floor $F$ | 95 000 EUR |
| Investment horizon $T$ | 2 years |
| Current stock price $S_0$ | 100 EUR |
| Continuously compounded annual interest rate $r$ | 0.10 |
| Annual stock volatility $\sigma$ | 0.30 |
| Dividends during time to maturity | |
| Case i): continuous dividends $d$ | 0.02 |
| Fall ii): dividends with present value $D_0$ | 5 EUR |

Table 2.7: Data of Example 2.5

floor $F$ and the capital invested $V$ the possibly negative minimum return of a one year investment is given by $\rho = \frac{F-V}{V}$.

Strategies 1 and 2 (described above) are illustrated in another example.

**Example 2.5**
*Starting with the hypothesis that the investor has decided to invest in stock, depending on the type of return of the object, we can distinguish two cases (for negative returns, as storage costs of real values for example, the approach can be applied analogously):*

**i)** *continuous dividend yield d*

**ii)** *ex ante known discrete yields with a time 0 discounted total value of* $D_0$.

*Data relating to the example is shown in Table 2.7. The volatility can be interpreted as a measure of variability of the stock price. The notion of volatility is an essential part of option pricing and will be looked at extensively later. Placing our considerations at the beginning,* $t = 0$ *of the investment, the time to maturity is* $\tau = T - t = T$. *For both strategies the goal is to determine the number n of stocks and/or (European) options and their delivery price K.*

**Case i)**
*The stock pays a continuous dividend at rate* $d = 2\%$ *p.a. which the investor reinvests immediately. At maturity T, the position in the stock has grown from n stocks to* $ne^{d\tau}$ *with* $\tau = T - 0 = T$. *Thus, for strategy 1 the investor needs to buy in* $t = 0$ *the same number of put options. Since the amount he*

*wants to invest in* $t = 0$ *is* $V$ *it must hold*

$$n \cdot S_0 + ne^{d\tau} \cdot P_{K,T}(S_0, \tau) = V. \tag{2.21}$$

*The investor chooses the put options delivery price* $K$ *such that his capital after two years does not drop below the floor he is aiming for* $F$*. That is, exercising the puts in time* $T$ *(if* $S_T \leq K$*) must give the floor* $F$ *which gives the second condition*

$$ne^{d\tau} \cdot K = F \iff n = \frac{F}{K} e^{-d\tau}. \tag{2.22}$$

*Substituting equation (2.22) into equation (2.21) gives*

$$e^{-d\tau} S_0 + P_{K,T}(S_0, \tau) - \frac{V}{F} \cdot K = 0. \tag{2.23}$$

*Due to the Black–Scholes pricing formula for European options that will be derived later in Section 6.2 the put price is expressed as a function of the parameter* $K$*. The delivery price which the investor is looking for can be computed by solving equation (2.23) numerically, for example by means of the Newton–Raphson method. In this case,* $K$ *is equal to* 99.56*. To be sure that the capital value does not drop below the floor* $F = 95\,000$ EUR *the investor buys* $n = \frac{F}{K} e^{-d\tau} = 916.6$ *stocks and* $n \cdot e^{d\tau} = 954$ *puts with delivery price* $K = 99.56$*. The price of the put option given by the Black–Scholes formula is* 8.72 EUR*/put.* SFEexerput

*Following the corresponding strategy* 2 *the investor invests* $Fe^{-r\tau} = 77\,779.42$ EUR *in bonds at time 0 which gives compounded to time* $T$ *exactly the floor* $F = 95\,000$ EUR*. For the remaining capital of* $V - Fe^{-r\tau} = 22\,220.58$ EUR *the investor buys* 954 *calls with delivery price* $K = 99.56$ *which has a price of* 23.29 EUR*/call according to the Black–Scholes formula. From the put–call parity follows the equivalence of both strategies, i.e. that both portfolios consisting of stocks and puts respectively zero bonds and calls have at each time* $t$ *the same value:*

$$n \cdot S_t + ne^{d\tau} P_{K,T}(S_t, \tau) = nKe^{-b\tau} + ne^{d\tau} C_{K,T}(S_0, \tau) \tag{2.24}$$

*where* $\tau = T - t, b = r - d$ *and* $nKe^{-b\tau} = Fe^{-r\tau}$ *due to equation (2.22). Table 2.8 shows the risk decreasing effect of the insurance.*

**Case ii)**

*Until maturity the stock pays dividends with a time 0 discounted total value* $D_0 = 5$ EUR *which, after distribution, are immediately invested in bonds. At time* $T$ *the dividend yield has a compounded value of* $D_T = D_0 e^{r\tau} = 6.107$ EUR *where* $\tau = T$ *denotes the time to maturity. Reasoning as in case i) and*

| Stock price $S_T$ [EUR] | Non–insured portfolio Value [EUR] | Return % p.a. | Insured portfolio Value [EUR] | Return % p.a. | Insured portfolio in % of the non–insured portfolio |
|---|---|---|---|---|---|
| 70 | 72857 | –27 | 95000 | –5 | 130 |
| 80 | 83265 | –17 | 95000 | –5 | 114 |
| 90 | 93673 | –6 | 95000 | –5 | 101 |
| 100 | 104081 | +4 | 95400 | –5 | 92 |
| 110 | 114489 | +15 | 104940 | +5 | 92 |
| 120 | 124897 | +25 | 114480 | +14 | 92 |
| 130 | 135305 | +35 | 124020 | +24 | 92 |
| 140 | 145714 | +46 | 133560 | +34 | 92 |

Table 2.8: The effect of a portfolio insurance in case i) on portfolio value and return. SFEoptman

*taking the dividend $D_T$ into account the investor buys $n$ stocks respectively $n$ puts, and obtains the following equations*

$$n \cdot S_0 + nP_{K,T}(S_0 - D_0, \tau) = V \tag{2.25}$$

*and*

$$nK + nD_T = F. \tag{2.26}$$

*The substraction of the cash dividend $D_0$ from the stock price $S_0$ in the option price formula cannot be justified until we introduce the binomial model (Chapter 7). Briefly, in a perfect market the stock price decreases instantaneously by the amount of the distributed dividend. Otherwise, an arbitrage opportunity arises. Substituting equation (2.26) into equation (2.25) gives:*

$$S_0 + P_{K,T}(S_0 - D_0, \tau) - \frac{V}{F} \cdot (K + D_T) = 0 \tag{2.27}$$

*Solving the equations analogously as in case i) the number $n$ of stocks and puts and the delivery price $K$ for strategy 1 are obtained:*

$$K = 96.42 \quad \textit{und} \quad n = \frac{F}{K + D_T} = 926.55$$

*For strategy 2 the investor buys* 926.55 *calls at a price of* 23.99 EUR*/call with a delivery price $K = 96.42$. He invests* $95\,000e^{-r\tau} = 77\,779.42$ *in bonds. For case ii) the effect of the portfolio insurance for both strategies is shown in Table 2.9 taking into account the time $T$ compounded total dividend.*

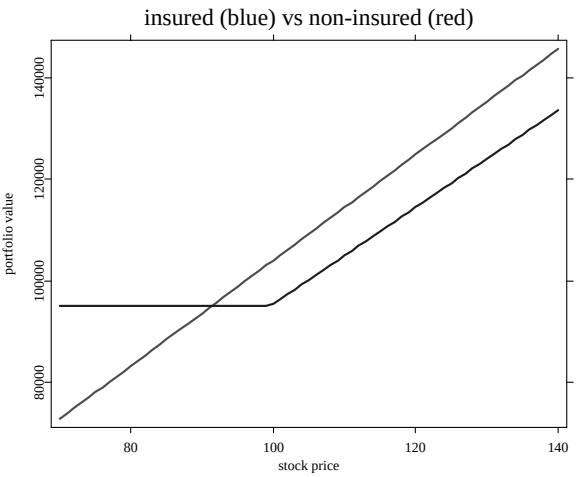



Figure 2.1: The effect of a portfolio insurance: While the straight line represents the value of the insured portfolio as a function of the stock price, the dotted line represents the value of the non–insured portfolio as a function of the stock price. SFEoptman

The example shows how a portfolio insurance can be carried out by means of options in principle. In practice, the following problems frequently occur:

- The number $n$ of stocks and options is not an integer. In a perfect financial market financial instruments are perfectly divisible, in reality, however, this is not the case. The error resulting from rounding up or down to the closest integer can be neglected only in large portfolios.
- Puts and calls traded on the market do not cover the whole range of delivery prices. Thus, options with the computed delivery price may not be available. Furthermore, options typically expire in less than one year which makes static strategies only viable for long–term investments.
- Finally, the market firstly provides American options, which are more expensive than European options, and which are sufficient to insure the portfolio. The additional exercise opportunities offered by American options, are only of interest if the investor might need to close the portfolio early.

The fact that options are not traded at all delivery prices suggests that they should be produced by the delta hedge process described in Chapter 6. How-

| Stock price $S_T$ [EUR] | Non–insured portfolio Value [EUR] | Return % p.a. | Insured portfolio Value [EUR] | Return % p.a. | Insured portfolio in % of the non–insured portfolio |
|---|---|---|---|---|---|
| 70 | 76107 | –24 | 94996 | –5 | 125 |
| 80 | 86107 | –14 | 94996 | –5 | 110 |
| 90 | 96107 | –4 | 94996 | –5 | 99 |
| 96.42 | 102527 | +3 | 94996 | –5 | 93 |
| 100 | 106107 | +6 | 98313 | –2 | 93 |
| 110 | 116107 | +16 | 107579 | +8 | 93 |
| 120 | 126107 | +26 | 116844 | +17 | 93 |
| 130 | 136107 | +36 | 126110 | +26 | 93 |
| 140 | 146107 | +46 | 135375 | +35 | 93 |

Table 2.9: The effect of a portfolio insurance in case ii) on portfolio value and return.

ever, since a dynamic strategy is involved, transaction costs need to be taken into account and give rise to other problems. Finally, it is worth pointing out that when insuring large portfolios it is convenient to hedge by means of index options, i.e. puts and calls on the DAX for example. This is not only beneficial from a cost saving point of view but also because index options replace options on a single underlying transaction which are not traded on the market. To compute the exact effect of an index option hedge, the correlation of the portfolio with the index is needed. The latter correlation is obtained from the correlations of each individual stock contained in the portfolio with the index. Besides detailed model assumptions such as the Capital Asset Pricing Model (CAPM see Section 11.4.1) which among others concern the stock, returns are required.

## 2.3 Binary One-Period Model

The simpliest of the option pricing formulae is the binomial option pricing formula. Here we take a look at a very simple model: *the binary one-period model*. The material in this section is only intended to be introductory. More details on the use of numerical procedures involving binomial trees are given in Chapter 7.

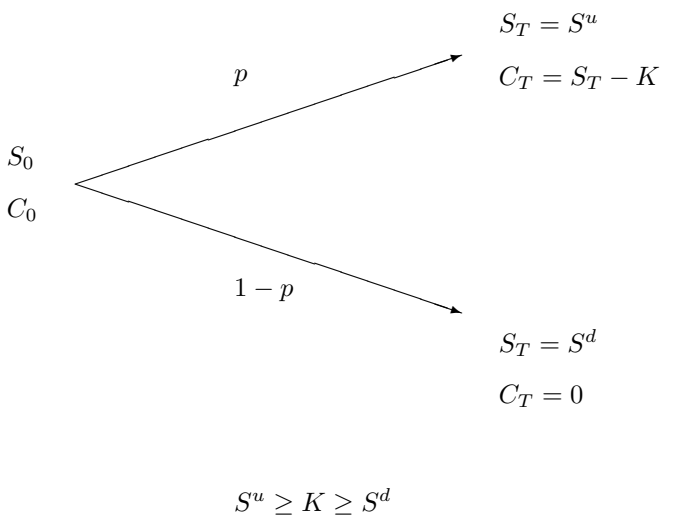



Figure 2.2: Stock and option prices in the binary one-period model.

Consider a stock with a price of $S_0$ and a European call option on the stock with a strike price $K$ where the current price is $C_0$. Assume that the call is being valued one period before expiration ($T$=1) and that the interest rate $r$ is equal to 0 in the one-period model. We let the future stock price be one of only two values: the stock price can either increase from $S_0$ to $S^u$ with probability $p$ or decrease from $S_0$ to $S^d$ with probability $(1-p)$. If the stock price moves up to $S^u$, the payoff will be $S_T - K$; if the stock price moves down to $S^d$, the payoff will be 0, see Figure 2.2.

Our goal is to determine the value $C_0$ of the call. The following different investment possibilities exist:

1. zerobond (with interest rate $r = 0$),
2. $S_0$ the current value of the stock,
3. $C_0$ ($C_T$) the price of European call at time 0 ($T$) with strike price $K$.

In order to value the call correctly, we examine two strategies. The first one is simply to buy the call. The second strategy is to choose a certain number of stocks $x$ and a decisive amount of a zerobond $y$ in a way that ensures the same payoff as the call at time $T$. Table 2.10 shows the cash flows for both strategies.

| Strategy | Flow at 0 | Flow at $T$ | |
|---|---|---|---|
| | | $S^u$ | $S^d$ |
| Call | $-C_0$ | $S_T - K$ | 0 |
| Stock + Zerobond | $-(xS^u + y)$ | $xS^u + y$ | $xS^d + y$ |

Table 2.10: Cash flow of call and portfolio of stock and zerobond.

In order to duplicate the payoff of the "buy-a-call" strategy, both cash flows must match whether the stock price goes up or down:

$$xS^u + y = r^u = S_T - K \tag{2.28}$$
$$xS^d + y = r^d = 0 \tag{2.29}$$

Since the desired payoff will be identical for both strategies, the second portfolio strategy is called *hedge strategy*. Solving (2.28) - (2.29) we obtain:

$$x = \frac{r^u - r^d}{S^u - S^d} \tag{2.30}$$
$$y = \frac{S^u r^d - S^d r^u}{S^u - S^d} \tag{2.31}$$

Now we have the situation that one gets the same financial product (payment of $r^u$ or $r^d$) for the price $C_0$ (option price) or for the price

$$xS_0 + y = S_0 \frac{r^u - r^d}{S^u - S^d} + \frac{S^u r^d - S^d r^u}{S^u - S^d} \tag{2.32}$$
$$= \frac{(S^u - S_0)r^d + (S_0 - S^d)r^u}{S^u - S^d} \tag{2.33}$$

Because the return of the option equals the return of the hedge strategy, both strategies must have the *same* initial cost:

$$xS_0 + y = C_0 \tag{2.34}$$

In fact, this yields the fair price $C_0$ of the call option:

$$C_0 = \frac{(S^u - S_0)r^d + (S_0 - S^d)r^u}{S^u - S^d} \tag{2.35}$$

It is worth noting that the price $C_0$ does not depend on $p$ (probability of rising stock). This means clearly that we do not need to regard the probabilties of future up and down movements, because they are already incorporated in the

current stock prices. In a two-period model the call price ultimately depends on the current stock price $S_0$, the possible value of the stock at time $T$ ($S^u$ and $S^d$) and the payoffs of the call at expiration ($r^u$ and $r^d$). Simplifying the last equation we obtain:

$$\begin{aligned} C_0 &= \frac{(S^u - S_0)r^d + (S_0 - S^d)r^u}{S^u - S^d} \\ &= \frac{S^u - S_0}{S^u - S^d} r^d + \frac{S_0 - S^d}{S^u - S^d} r^u \\ &= (1-q)r^d + qr^u \end{aligned} \tag{2.36}$$

with

$$q = \frac{S_0 - S^d}{S^u - S^d}. \tag{2.37}$$

Another way to look at (2.36) is to say that $C_0$ is the expected value of $C_T$ with respect to the probability measure defined by $q$. This leads us to a second approach based on the *martingale model*. A martingale is a stochastic process where the expected value of all future values equals the current value. In the binary one-period model a martingale measure is simply given by a probability measure $q$ such that the expected return of the share at time $T$ is 0, i.e. the expected future stock price equals the current stock price $S_0$ (for interest rate $r = 0$):

$$S_0 = \mathsf{E}_q(S_T) = (1-q)S^d + qS^u \tag{2.38}$$

where $\mathsf{E}_q$ indicates an expectation taken with respect to the martingale measure $q$. In terms of the one-period binomial tree, $q$ is the probability of an "up" move and $(1-q)$ the probability of a "down" move, see Figure 2.3.

Solving (2.38), we obtain:

$$q = \frac{S_0 - S^d}{S^u - S^d}. \tag{2.39}$$

The martingale approach is now to determine the price $C_0$ as the expected payoff

$$\mathsf{E}_q\left[\max\left\{S_T - K, 0\right\}\right] = \mathsf{E}_q\left[C_T\right]. \tag{2.40}$$

Calculating this expected value of $C_T$, we derive the price of the call option:

$$C_0 = (1-q)r^d + qr^u. \tag{2.41}$$

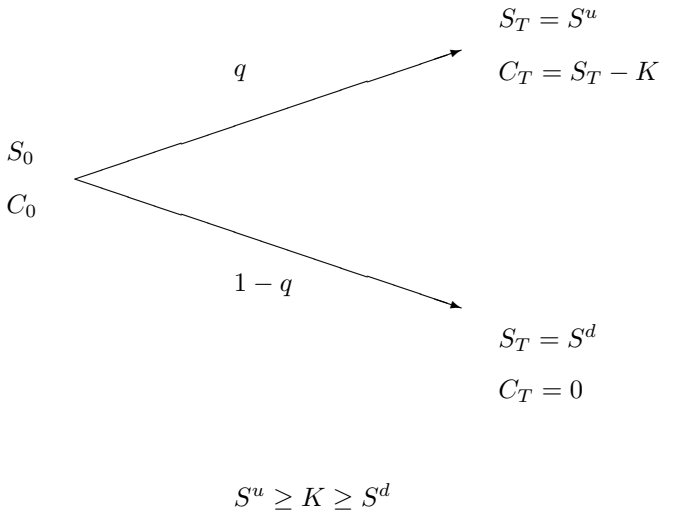


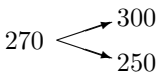

Figure 2.3: Stock and option prices in the martingale model

This is the same value we concluded from the former duplicating portfolio approach. In the case of interest rate $r > 0$ the option price changes to

$$C_0 \quad = \quad (1+r)^{-1} \mathsf{E}_q \left[ C_T \right]. \tag{2.42}$$

The martingale measure approach is analogous to Cox and Ross (1976) *risk neutral valuation* that one finds in for example Hull (2006).

**Example 2.6**
*Consider the following binary one-period model: a stock with price* $S_0 = 270$*, a European call option on a stock with strike* $K = 270$*, a zerobond with price* $1 \cdot (1+r)^{-1}$ *with interest rate* $r = 5\%$ *(i.e. price of zerobond corresponds to* $1/1.05$*). The stock can either increase to* $300$ *or decrease to* $250$*:*




*In the Hedge portfolio approach on calculates:*
*(1)* $S_T = 300$, $C_T = 30$ *i.e.* $300x + y = 30$

*(2)* $S_T = 250$, $C_T = 0$ *i.e.* $250x + y = 0$
*Solving this linear system we obtain* $x = 0.6$, $y = -150$. *Thus, selling* 150 *bonds and buying* 0.6 *stocks duplicates the payoff of the call. Therefore the price of the call is:*

$$\begin{aligned} xS_0 + y &= C_0 \\ 0.6 \cdot 270 + (-150) \cdot \frac{1}{1.05} &= 19.14 \end{aligned}$$

*and the corresponding probability is* $q = \frac{270-250}{300-250} = 0.4$.

*In the martingale measure approach* $S$ *has to be a martingale under the measure* $q$, *i.e. the expected value of the discounted process* $S_T/(1+r)$ *equals* $S_0$*:*

$$S_0 = 270 = \mathsf{E}_q\left[\frac{S_T}{1+r}\right] = q \cdot \frac{300}{1.05} + (1-q) \cdot \frac{250}{1.05}$$

*This leads to* $q = 0.67$. *The price of the call is therefore*

$$\begin{aligned} C_0 &= \mathsf{E}_q\left[\frac{C_T}{1+r}\right] \\ &= q \cdot \frac{30}{1.05} + (1-q) \cdot \frac{0}{1.05} \\ &= 0.67 \cdot \frac{30}{1.05} \\ &= 19.14 \end{aligned}$$

*We can see that both approaches provide the same results.*

We see from the previous example that there are two different approaches of calculation for the option price:

- calculation of the hedge strategy, i.e. determine portfolio of stocks and bonds that duplicates the cash flow of the option;
- martingale measure approach, i.e. calculation of the expected option payment in the martingale model.

## 2.4 Recommended Literature

The fundamental arbitrage relations of financial derivatives can be found in every modern finance textbook, as for example Hull (2000) and Hull (2006).

In principle each option pricing theory is based on theses relations, as it is the case for the model of Black and Scholes (1973) for example, see also the fundamental article of Merton (1973). The idea of portfolio assurance was introduced by Leland (1980). It is nowadays covered above all by practical risk management textbooks as Das (1997). The hedge portfolio approach and the martingale approach used in the binary one-period model for calculation of option prices are discussed in Elton, Gruber, Brown and Goztmann (2002) and Ross, Westerfield and Jaffe (2005). The martingale measure approach analogous to the classic work of Cox and Ross (1976) is presented in Yan (1999) and Hull (2006).

# 3 Basic Concepts of Probability Theory

## 3.1 Real Valued Random Variables

Thanks to Newton's laws, dropping a stone from a hight of 10 m, the point of time of its impact on the ground is known before executing the experiment. Quantities in complex systems (such as stock prices at a certain date, daily maximum temperature at a certain place) are , however, not deterministically predictable, although it is known which values are more likely to occur than others. Contrary to the falling stone, data which cannot be described successfully by deterministic mechanism, can be modelled by random variables.

Let $X$ be such a random variable that (as a model for stock prices) takes values on real time. The appraisal of which values of $X$ are more and which are less likely, is expressed by the *probability* of events as $\{a < X < b\}$ or $\{X \leq b\}$. The set of all probabilities

$$\mathrm{P}(a \leq X \leq b) \ , \quad -\infty < a \leq b < \infty \ ,$$

determines the *distribution* of $X$. In other words, the distribution is defined by the probabilities of all events which depend on $X$. In the following, we denote the probability distribution of $X$ by $\mathcal{L}(X)$.

The probability distribution is uniquely defined by the *cumulative probability distribution*

$$F(x) = \mathrm{P}(X \leq x) \ , \quad -\infty < x < \infty \ .$$

$F(x)$ monotonously increases and converges for $x \to -\infty$ to 0, and for $x \to \infty$ to 1. If there is a function $p$, such that the probabilities can be computed by means of an integral

$$\mathrm{P}(a < X < b) = \int_a^b p(x)dx \, ,$$

$p$ is called a *probability density*, or briefly density of $X$. Then the cumulative

distribution function is a primitive of $p$:

$$F(x) = \int_{-\infty}^{x} p(y)dy.$$

For small $h$ it holds:

$$\mathrm{P}(x - h < X < x + h) \approx 2h \cdot p(x).$$

Thus $p(x)$ is a measure of the likelihood that $X$ takes values close to $x$.

The most important family of distributions with densities, is the *normal distribution* family. It is characterised by two parameters $\mu, \sigma^2$. The densities are given by

$$\varphi_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sigma}\,\varphi\big(\frac{x-\mu}{\sigma}\big)\,,$$

$$\varphi(x) = \varphi_{0,1}(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}\,.$$

The distribution with density $\varphi(x)$ is called *standard normal distribution*. "$X$ a normal random variable with parameters $\mu, \sigma^2$" is commonly abbreviated by "$X$ is $N(\mu, \sigma^2)$ distributed". The cumulative distribution function of a standard normal distribution is denoted by $\Phi$ and it holds:

$$\Phi(x) = \int_{-\infty}^{x} \varphi(y)dy.$$

If $X$ is $N(\mu, \sigma^2)$ distributed, then the centered and scaled random variable $(X - \mu)/\sigma$ is standard normal distributed. Therefore, its cumulative distribution function is given by:

$$F(x) = \mathrm{P}(X \le x) = \mathrm{P}\big(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\big) = \Phi\big(\frac{x-\mu}{\sigma}\big).$$

A distribution which is of importance in modelling stock prices is the *lognormal distribution*. Let $X$ be a positive random variable whose natural logarithm $\log(X)$ is normally distributed with parameters $\mu, \sigma^2$. We say that $X$ is lognormally distributed with parameters $\mu, \sigma^2$. Its cumulative distribution function follows directly from the above definition:

$$F(x) = \mathrm{P}(X \le x) = \mathrm{P}(\log X \le \log x) = \Phi\big(\frac{\log x - \mu}{\sigma}\big), \quad x > 0.$$

Deriving $F(x)$ once, we obtain its density function with parameters $\mu, \sigma^2$:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\,\frac{1}{x}\, e^{-\frac{(\log x-\mu)^2}{2\sigma^2}} = \frac{1}{\sigma x}\,\varphi\big(\frac{\log x - \mu}{\sigma}\big), \quad x > 0.$$

If $X$ is a random variable that takes only finite or countably infinite values $x_1, \ldots, x_n$, then $X$ is said to be a discrete random variable and its distribution is fully determined by the probabilities:

$$\mathrm{P}(X = x_k)\ , \quad k = 1, \ldots, n.$$

The simplest discrete random variables take only 2 or 3 values, for example $\pm 1$ or $-1, 0, +1$. They constitute the basis of binomial or trinomial trees which can be used to construct discrete random processes in computers. Such tree methods are reasonable approximations to continuous processes which are used to model stock prices.

In this context, *binomially distributed* random variables appear. Let $Y_1, \ldots, Y_n$ be independent random variables taking two values, 0 or 1, with probabilities

$$p = \mathrm{P}(Y_k = 1)\ ,\ 1 - p = \mathrm{P}(Y_k = 0)\ , \quad k = 1, \ldots, n\,.$$

We call such random variables *Bernoulli distributed* with parameter $p$. The number of ones appearing in the sample $Y_1, \ldots, Y_n$, equals the sum $X = \sum_{k=1}^{n} Y_k$ which is binomial distributed with parameters $n, p$ :

$$X = \sum_{k=1}^{n} Y_k\ ,\ \mathrm{P}(X = m) = \binom{n}{m} p^m (1-p)^{n-m}\,, \quad m = 0, \ldots, n\,.$$

SFEBinomial

Instead of saying $X$ is binomial distributed with parameters $n, p$, we use the notation "$X$ is $B(n, p)$ distributed". Hence, a Bernoulli distributed random variable is $B(1, p)$ distributed.

If $n$ is large enough, a $B(n, p)$ distributed random variable can be approximated by a $N(np,\, np(1-p))$ distributed random variable $Z$, in the sense that

$$\mathrm{P}(a < X < b) \approx \mathrm{P}(a < Z < b)\,. \tag{3.1}$$

The central limit theorem is more precise on that matter. In classical statistics it is used to avoid, for large $n$, the tedious calculation of binomial probabilities. Conversely, it is possible to approximate the normal distribution by an easy simulated binomial tree. SFEclt

## 3.2 Expectation and Variance

The mathematical *expectation* or the *mean* $\mathsf{E}[X]$ of a real random variable $X$ is a measure for the location of the distribution of $X$. Adding to $X$ a real

constant $c$, it holds for the expectation: $\mathsf{E}[X+c] = \mathsf{E}[X]+c$, i.e. the location of the distribution is translated. If $X$ has a density $p(x)$, its expectation is defined as:

$$\mathsf{E}(X) = \int_{-\infty}^{\infty} xp(x)dx.$$

If the integral does not exist, neither does the expectation. In practice, this is rather rarely the case.

Let $X_1, \ldots, X_n$ be a sample of identically independently distributed (i.i.d.) random variables (see Section 3.4) having the same distribution as $X$, then $\mathsf{E}[X]$ can be estimated by means of the sample mean:

$$\hat{\mu} = \frac{1}{n}\sum_{t=1}^{n} X_t.$$

A measure for the dispersion of a random variable $X$ around its mean is given by the *variance* $\mathrm{Var}(X)$:

$$\begin{array}{rcl} \mathrm{Var}(X) & = & \mathsf{E}[(X - \mathsf{E}\,X)^2] \\ \text{Variance} & = & \text{mean squarred deviation of a random variable} \\ & & \text{around its expectation.} \end{array}$$

If $X$ has a density $p(x)$, its variance can be computed as follows:

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mathsf{E}\,X)^2 p(x)dx.$$

The integral can be infinite. There are empirical studies giving rise to doubt that some random variables appearing in financial and actuarial mathematics and which model losses in highly risky businesses dispose of a finite variance.

As a quadratic quantity the variance's unity is different from that of $X$ itself. It is better to use the standard deviation of $X$ which is measured in the same unity as $X$:

$$\sigma(X) = \sqrt{\mathrm{Var}(X)}\,.$$

Given a sample of i.i.d. variables $X_1, \ldots, X_n$ which have the same distribution as $X$, the sample variance can be estimated by:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{t=1}^{n}(X_t - \hat{\mu})^2.$$

A $N(\mu, \sigma^2)$ distributed random variable $X$ has mean $\mu$ and variance $\sigma^2$. The $2\sigma$ area around $\mu$ contains with probability of a little more than 95% observations of $X$:

$$\mathrm{P}(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95\,.$$

A lognormally distributed random variable $X$ with parameters $\mu$ and $\sigma^2$ has mean and variance

$$\mathsf{E}(X) = e^{\mu + \frac{1}{2}\sigma^2}, \quad \mathrm{Var}(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).$$

A $B(n, p)$ distributed variable $X$ has mean $np$ and variance $np(1-p)$. The approximation (3.1) is chosen such that the binomial and normal distribution have identical mean and variance.

## 3.3 Skewness and Kurtosis

**Definition 3.1 (Skewness)**
*The skewness of a random variable $X$ with mean $\mu$ and variance $\sigma^2$ is defined as*

$$S(X) = \frac{\mathsf{E}[(X-\mu)^3]}{\sigma^3}.$$

If the skewness is negative (positive) the distribution is skewed to the left (right). Normally distributed random variables have a skewness of zero since the distribution is symmetrical around the mean. Given a sample of i.i.d. variables $X_1, \ldots, X_n$, Skewness can be estimated by (see Section 3.4)

$$\hat{S}(X) = \frac{\frac{1}{n}\sum_{t=1}^{n}(X_t - \hat{\mu})^3}{\hat{\sigma}^3}, \tag{3.2}$$

with $\hat{\mu}, \hat{\sigma}^2$ as defined in the previous section.

**Definition 3.2 (Kurtosis)**
*The kurtosis of a random variable $X$ with mean $\mu$ and variance $\sigma^2$ is defined as*

$$\mathrm{Kurt}(X) = \frac{\mathsf{E}[(X-\mu)^4]}{\sigma^4}.$$

Normally distributed random variables have a kurtosis of 3. Financial data often exhibits higher kurtosis values, indicating that values close to the mean

and extreme positive and negative outliers appear more frequently than for normally distributed random variables. Kurtosis can be estimated by

$$\widehat{\mathrm{Kurt}}(X) = \frac{\frac{1}{n}\sum_{t=1}^{n}(X_t - \hat{\mu})^4}{\hat{\sigma}^4}. \tag{3.3}$$

**Example 3.1** *The empirical standard deviation of monthly DAX data from 1979: 1 to 2000: 10 is* $\hat{\sigma} = 0.056$*, which corresponds to a yearly volatility of* $\hat{\sigma} \cdot \sqrt{12} = 0.195$*. Later in Section(6.4.5), we will explain the factor* $\sqrt{12}$ *in detail. The kurtosis of the data is equal to 5.428, i.e. greater than 3 which suggests a non-normality of the DAX returns.*

*SFEsumm*

## 3.4 Random Vectors, Dependence, Correlation

A *random vector* $(X_1, \ldots, X_n)$ from $\mathbb{R}^n$ can be useful in describing the mutual dependencies of several random variables $X_1, \ldots, X_n$, for example several underlying stocks. The joint distribution of the random variables $X_1, \ldots, X_n$ is as in the univariate case, uniquely determined by the probabilities

$$\mathrm{P}(a_1 \le X_1 \le b_1, \ldots, a_n \le X_n \le b_n)\ , \quad -\infty < a_i \le b_i < \infty\ ,\ i = 1, ..., n\,.$$

If the random vector $(X_1, \ldots, X_n)$ has a density $p(x_1, \ldots, x_n)$, the probabilities can be computed by means of the following integrals:

$$\mathrm{P}(a_1 \le X_1 \le b_1, \ldots, a_n \le X_n \le b_n) = \int_{a_n}^{b_n} \ldots \int_{a_1}^{b_1} p(x_1, \ldots, x_n) dx_1 \ldots dx_n.$$

The univariate or marginal distribution of $X_j$ can be computed from the joint density by integrating out the variable not of interest.

$$\mathrm{P}(a_j \le X_j \le b_j) = \int_{-\infty}^{\infty} \ldots \int_{a_j}^{b_j} \ldots \int_{-\infty}^{\infty} p(x_1, \ldots, x_n) dx_1 \ldots dx_n.$$

The intuitive notion of *independence* of two random variables $X_1, X_2$ is formalised by requiring:

$$\mathrm{P}(a_1 \le X_1 \le b_1,\ a_2 \le X_2 \le b_2) = \mathrm{P}(a_1 \le X_1 \le b_1) \cdot \mathrm{P}(a_2 \le X_2 \le b_2),$$

i.e. the joint probability of two events depending on the random vector $(X_1, X_2)$ can be factorised. It is sufficient to consider the univariate distributions of $X_1$ and $X_2$ exclusively. If the random vector $(X_1, X_2)$ has a

density $p(x_1, x_2)$, then $X_1$ and $X_2$ have densities $p_1(x)$ and $p_2(x)$ as well. In this case, independence of both random variables is equivalent to a joint density which can be factorised:

$$p(x_1, x_2) = p_1(x_1)p_2(x_2).$$

Dependence of two random variables $X_1, X_2$ can be very complicated. If $X_1, X_2$ are jointly normally distributed, their dependency structure can be rather easily quantified by their covariance:

$$\mathrm{Cov}(X_1, X_2) = \mathsf{E}[(X_1 - \mathsf{E}[X_1])(X_2 - \mathsf{E}[X_2])],$$

as well as by their correlation:

$$\mathrm{Corr}(X_1, X_2) = \frac{\mathrm{Cov}(X_1, X_2)}{\sigma(X_1) \cdot \sigma(X_2)}\,.$$

The correlation has the advantage of taking values between -1 and +1, which is scale invariant. For jointly normally distributed random variables, independence is equivalent to zero correlation, while complete dependence is equivalent to either a correlation of +1 ($X_1$ is large when $X_2$ is large) or a correlation of -1 ($X_1$ is large when $X_2$ is small).

In general, it holds for *independent* random variables $X_1, \ldots, X_n$

$$\mathrm{Cov}(X_i, X_j) = 0 \qquad \text{for } i \neq j\,.$$

This implies a useful computation rule:

$$\mathrm{Var}\big(\sum_{j=1}^{n} X_j\big) = \sum_{j=1}^{n} \mathrm{Var}(X_j)\,.$$

If $X_1, \ldots, X_n$ are independent and have all the same distribution:

$$\mathrm{P}(a \leq X_i \leq b) = \mathrm{P}(a \leq X_j \leq b) \qquad \text{for all } i, j\,,$$

we call them *independently and identically distributed (i.i.d.)*.

## 3.5 Conditional Probabilities and Expectations

The *conditional probability* that a random variable $Y$ takes values between $a$ and $b$ conditioned on the event that a random variable $X$ takes values between $x$ and $x + \Delta_x$, is defined as

$$\mathrm{P}(a \leq Y \leq b | x \leq X \leq x + \Delta_x) = \frac{\mathrm{P}(a \leq Y \leq b\,,\, x \leq X \leq x + \Delta_x)}{\mathrm{P}(x \leq X \leq x + \Delta_x)}\,, \quad (3.4)$$

provided the denominator is different from zero. The conditional probability of events of the kind $a \leq Y \leq b$ reflects our opinion of which values are more plausible than others, given that another random variable $X$ has taken a certain value. If $Y$ is independent of $X$, the probabilities of $Y$ are not influenced by prior knowledge about $X$. It holds:

$$\mathrm{P}(a \leq Y \leq b | x \leq X \leq x + \Delta x) = \mathrm{P}(a \leq Y \leq b)\,.$$

As $\Delta x$ goes to 0 in equation (3.4), the left side of equation (3.4) converges heuristically to $\mathrm{P}(a \leq Y \leq b | X = x)$. In the case of a continuous random variable $X$ having a density $p_X$, the left side of equation (3.4) is not defined since $\mathrm{P}(X = x) = 0$ for all $x$. But, it is possible to give a sound mathematical definition of the conditional distribution of $Y$ given $X = x$. If the random variables $Y$ and $X$ have a joint distribution $p(x, y)$, then the conditional distribution has the density

$$p_{Y|X}(y|x) = \frac{p(x,y)}{p_X(x)} \quad \text{for} \quad p_X(x) \neq 0$$

and $p_{Y|X}(y|x) = 0$ otherwise. Consequently, it holds:

$$\mathrm{P}(a \leq Y \leq b | X = x) = \int_a^b p_{Y|X}(y|x)dy.$$

The expectation with respect to the conditional distribution can be computed by:

$$\mathsf{E}(Y|X = x) = \int_{-\infty}^{\infty} y\, p_{Y|X}(y|x)dy \stackrel{\text{def}}{=} \eta(x).$$

The function $\eta(x) = \mathsf{E}(Y|X = x)$ is called the *conditional expectation of* $Y$ *given* $X = x$. Intuitively, it is the expectation of the random variable $Y$ knowing that $X$ has taken the value $x$.

Considering $\eta(x)$ as a function of the random variable $X$ the conditional expectation of $Y$ given $X$ is obtained:

$$\mathsf{E}(Y|X) = \eta(X).$$

$\mathsf{E}(Y|X)$ is a random variable, which can be regarded as a function having the same expectation as $Y$. The conditional expectation has some useful properties, which we summarise in the following theorem.

**Theorem 3.1** *Let $X, Y, Z$ be real valued continuous random variables having a joint density.*

*a) If $X, Y$ are independent, then* $\mathsf{E}(Y|X = x) = \mathsf{E}(Y)$

*b) If $Y = g(X)$ is a function of $X$, then*

$$\mathsf{E}[Y|X = x] = \mathsf{E}[g(X)|X = x] = g(x).$$

*In general, it holds for random variables of the kind $Y = Zg(X)$ :*

$$\mathsf{E}[Y|X = x] = \mathsf{E}[Zg(X)|X = x] = g(x)\,\mathsf{E}[Z|X = x].$$

*c) The conditional expectation is linear, i.e. for any real numbers $a, b$ it holds:*

$$\mathsf{E}(aY + bZ|X = x) = a\,\mathsf{E}(Y|X = x) + b\,\mathsf{E}(Z|X = x).$$

*d) The law of iterated expectations:* $\mathsf{E}[\mathsf{E}(Y|X)] = \mathsf{E}(Y)$.

The concept of the conditional expectation can be generalised analogously for multivariate random vectors $Y$ and $X$. Let $S_t, t = 0, 1, 2, ...$ be a sequence of chronologically ordered random variables, for instance as a model of daily stock prices, let $Y = S_{t+1}$ and $X = (S_t, ..., S_{t-p+1})^\top$, then the conditional expectation

$$\mathsf{E}(Y|X = x) = \mathsf{E}(S_{t+1}|S_t = x_1, ..., S_{t-p+1} = x_p)$$

represents the expected stock price of the following day $t + 1$ given the stock prices $x = (x_1, ..., x_p)^\top$ of the previous $p$ days. Since the information available at time $t$ (relevant for the future evolution of the stock price) can consist of more than only a few past stock prices, we make frequent use of the notation $\mathsf{E}(Y|\mathcal{F}_t)$ for the expectation of $Y$ given the information available up to time $t$. For all $t$, $\mathcal{F}_t$ denotes a family of events (having the structure of a $\sigma$–algebra, i.e. certain combinations of events of $\mathcal{F}_t$ are again elements of $\mathcal{F}_t$) representing the information available up to time $t$. $\mathcal{F}_t$ consists of events of which it is known whether they occur up to time $t$ or not. Since more information unveils as time evolves, we must have $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$, see Definition 5.1. Leaving out the exact definition of $\mathsf{E}(Y|\mathcal{F}_t)$ we confine to emphasise that the computation rules given in Theorem 3.1, appropriately reformulated, can be applied to the general conditional expectation.

## 3.6 Recommended Literature

Durrett (1991), Ross (1994), Pitman (1997), Krengel (2000) and Jacod and Protter (2000), among others, give an introduction to probability theory. An introduction to martingale theory which is imperative for the understanding of advanced mathematical finance is given by Williams (1991).

# 4 Stochastic Processes in Discrete Time

A *stochastic process* or random process consists of chronologically ordered random variables $\{X_t;\ t \geq 0\}$. For simplicity we assume that the process starts at time $t = 0$ in $X_0 = 0$. In this chapter, we consider exclusively processes in *discrete time*, i.e. processes which are observed at equally spaced points time $t = 0, 1, 2, \ldots$ . Typical examples are daily, monthly or yearly observed economic data as stock prices, rates of unemployment or sales figures.

## 4.1 Binomial Processes

One of the simplest stochastic processes is an *ordinary random walk*, a process whose increments $Z_t = X_t - X_{t-1}$ from time $t-1$ to time $t$ take exclusively the values +1 or -1. Additionally, we assume the increments to be i.i.d. and independent of the starting value $X_0$. Hence, the ordinary random walk can be written as:

$$X_t = X_0 + \sum_{k=1}^{t} Z_k \qquad , \ t = 1, 2, \ldots \tag{4.1}$$

$X_0, Z_1, Z_2, \ldots$ independent and

$$\mathrm{P}(Z_k = 1) = p \quad , \quad \mathrm{P}(Z_k = -1) = 1 - p \qquad \text{for all } \ k\,.$$

Letting the process go up by $u$ and go down by $d$, instead, we obtain a more general class of *binomial processes*:

$$\mathrm{P}(Z_k = u) = p\,, \quad \mathrm{P}(Z_k = -d) = 1 - p \quad \text{for all } \ k\,,$$

where $u$ and $d$ are constant ($u$=up, $d$=down).

Linear interpolation of the points $(t, X_t)$ reflects the time evolution of the process and is called a *path* of an ordinary random walk. Starting in $X_0 = a$, the process moves on the grid of points $(t, b_t)$, $t = 0, 1, 2, \ldots,\ b_t = a - t,\ a - t + 1, \ldots, a + t$. Up to time $t$, $X_t$ can grow at most up to $a + t$ (if

$Z_1 = \ldots = Z_t = 1$) or can fall at least to $a - t$ (if $Z_1 = \ldots = Z_t = -1$). Three paths of an ordinary random walk are shown in Figure 4.1 ($p = 0.5$), 4.2 ($p = 0.4$) and Figure 4.3 ($p = 0.6$).

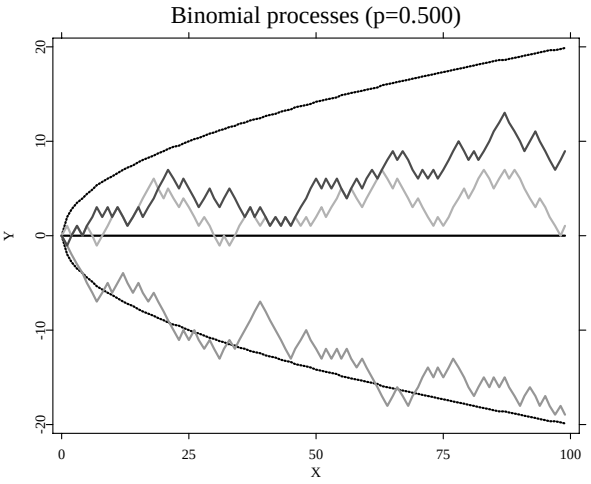



Figure 4.1: Three paths of a symmetric ordinary random walk. $(2\sigma)$–intervals around the drift (which is zero) are given as well. SFEBinomp

For generalised binomial processes the grid of possible paths is more complicated. The values which the process $X_t$ starting in $a$ can possibly take up to time $t$ are given by

$$b_t = a + n \cdot u - m \cdot d\,, \text{ where } n, m \geq 0\,, \quad n + m = t\,.$$

If, from time 0 to time $t$, the process goes up $n$ times and goes down $m$ times then $X_t = a + n \cdot u - m \cdot d$. That is, $n$ of $t$ increments $Z_1, \ldots, Z_t$ take the value $u$, and $m$ increments take the value $-d$. The grid of possible paths is also called a binomial tree.

The mean of the *symmetric ordinary random walk* ($p = \frac{1}{2}$) starting in 0 ($X_0 = 0$) is for all times $t$ equal to 0 :

$$\mathsf{E}[X_t] = 0 \qquad \text{for all } t\,.$$

Otherwise, the *random walk* has a *trend* or *drift*, for ($p > \frac{1}{2}$) it has a positive drift and for ($p < \frac{1}{2}$) it has a negative drift. The process grows or falls in

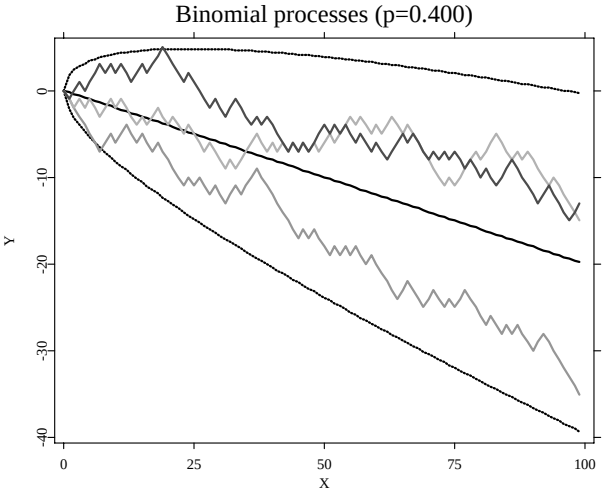



Figure 4.2: Three paths of an ordinary random walk with $p = 0.4$. $(2\sigma)$–intervals around the drift (which is the line with negative slope) are given as well. SFEBinomp

average:

$$\mathsf{E}[X_t] = t \cdot (2p - 1)\,,$$

since it holds for all increments $\mathsf{E}[Z_k] = 2p - 1$. Hence, the trend is linear in time. It is the upward sloping line in Figure 4.3 ($p = 0.6$) and the downward sloping line in Figure 4.2 ($p = 0.4$).

For the generalized *binomial process* with general starting value $X_0$ it holds analogously $\mathsf{E}[Z_k] = (u + d)p - d$ and thus:

$$\mathsf{E}[X_t] = \mathsf{E}[X_0] + t \cdot \{(u + d)p - d\}\,.$$

As time evolves the set of values $X_t$ grows, and its variability increases. Since the summands in (4.1) are independent and $\mathrm{Var}(Z_k) = \mathrm{Var}(Z_1)$ for all $k$, the variance of $X_t$ is given by (refer to Section 3.4):

$$\mathrm{Var}(X_t) = \mathrm{Var}(X_0) + t \cdot \mathrm{Var}(Z_1)\,.$$

Hence, the variance of $X_t$ grows linearly with time, as does the standard deviation. For the random walks depicted in Figure 4.1 ($p = 0.5$), Figure 4.2 ($p = 0.4$) and Figure 4.3 ($p = 0.6$) the intervals $[\mathsf{E}[X_t] - 2\sigma(X_t); \mathsf{E}[X_t] +$

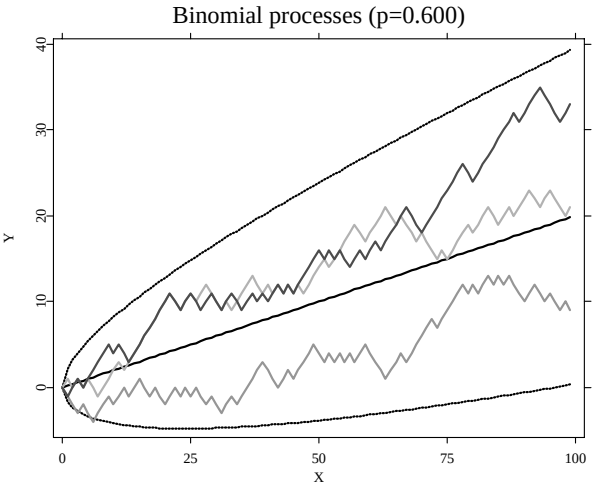



Figure 4.3: Three paths of an ordinary random walk with $p = 0.6$. $(2\sigma)$–intervals around the drift (which is the line with positive slope) are given as well. SFEBinomp

$2\sigma(X_t)]$ are shown as well. For large $t$ , these intervals should contain 95% of the realisations of processes.

The variance of the increments can be easily computed. We use the following result which holds for the binomial distribution. Define

$$Y_k = \frac{Z_k + d}{u + d} = \begin{cases} 1 & \text{if } Z_k = u \\ 0 & \text{if } Z_k = -d \end{cases}$$

or

$$Z_k = (u + d)\, Y_k - d \tag{4.2}$$

we obtain the following representation of the binomial process

$$X_t = X_0 + (u + d)\, B_t - td \tag{4.3}$$

where

$$B_t = \sum_{k=1}^{t} Y_k \tag{4.4}$$

is a $B(t, p)$ distributed random variable.

Given the distribution of $X_0$, the distribution of $X_t$ is specified for all $t$. It can be derived by means of a simple transformation of the binomial distribution $B(t,p)$. From equations (4.2) to (4.4) we obtain for $X_0 = 0$:

$$\mathrm{Var}(X_t) = t(u+d)^2\, p(1-p)$$

and for large $t$ the distribution of $X_t$ can be approximated by:

$$\mathcal{L}(X_t) \approx N(t\{(u+d)p - d\},\, t(u+d)^2\, p(1-p)).$$

For $p = \frac{1}{2}, u = d = \Delta x$, the following approximation holds for $\mathcal{L}(X_t)$:

$$N(0, t \cdot (\Delta x)^2).$$

Figure 4.4 shows the fit for $t = 100$.

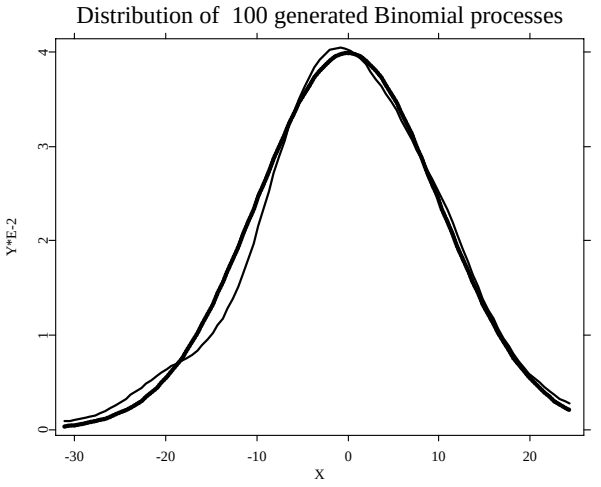



Figure 4.4: The distribution of 100 paths of an ordinary symmetric random walk of length 100 and a kernel density estimation of 100 normally distributed random variables. SFEbinomv

## 4.2 Trinomial Processes

In contrast to binomial processes, a *trinomial process* allows a quantity to stay constant within a given period of time. In the latter case, the increments are

described by:

$$\mathrm{P}(Z_k = u) = p\,,\ \mathrm{P}(Z_k = -d) = q\,,\ \mathrm{P}(Z_k = 0) = r = 1 - p - q\ ,$$

and the process $X_t$ is again given by:

$$X_t = X_0 + \sum_{k=1}^{t} Z_k$$

where $X_0, Z_1, Z_2, \ldots$ are mutually independent. To solve the Black–Scholes equation, some algorithms use trinomial schemes with time and state dependent probabilities $p$, $q$ and $r$. Figure 4.5 shows five simulated paths of a trinomial process with $u = d = 1$ and $p = q = 0.25$.

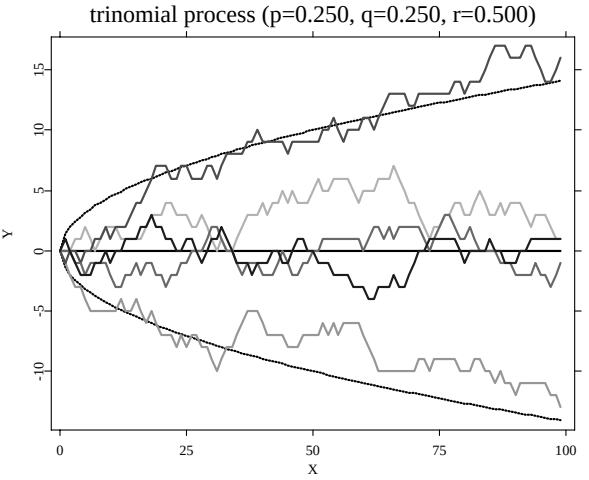



Figure 4.5: Five paths of a trinomial process with $p = q = 0.25$. $(2\sigma)$–intervals around the trend (which is zero) are given as well. SFETrinomp

The exact distribution of $X_t$ cannot be derived from the binomial distribution but for the trinomial process a similar relations hold:

$$\begin{aligned} \mathsf{E}[X_t] &= \mathsf{E}[X_0] + t \cdot \mathsf{E}[Z_1] = \mathsf{E}[X_0] + t \cdot (pu - qd) \\ \mathrm{Var}(X_t) &= \mathrm{Var}(X_0) + t \cdot \mathrm{Var}(Z_1), \text{ where} \\ \mathrm{Var}(Z_1) &= p(1-p)u^2 + q(1-q)d^2 + 2pq\ ud\ . \end{aligned}$$

For large $t$, $X_t$ is approximately $\mathrm{N}(\mathsf{E}[X_t],\ \mathrm{Var}(X_t))$–distributed.

## 4.3 General Random Walks

Binomial and trinomial processes are simple examples for *general random walks*, i.e. stochastic processes $\{X_t;\, t \geq 0\}$ satisfying:

$$X_t = X_0 + \sum_{k=1}^{t} Z_k\,, \qquad t = 1, 2, \ldots$$

where $X_0$ is independent of $Z_1, Z_2, \ldots$ which are i.i.d. The increments have a distribution of a real valued random variable. $Z_k$ can take a finite or countably infinite number of values; but it is also possible for $Z_k$ to take values out of a continuous set.

As an example, consider a *Gaussian random walk* with $X_0 = 0$, where the finitely many $X_1, \ldots, X_t$ are jointly normally distributed. Such a random walk can be constructed by assuming identically, independently and normally distributed increments. By the properties of the normal distribution, it follows that $X_t$ is $N(\mu t,\, \sigma^2 t)$–distributed for each $t$. If $X_0 = 0$ and $\mathrm{Var}(Z_1)$ is finite, it holds approximately for all random walks for $t$ large enough:

$$\mathcal{L}(X_t) \approx N(t \cdot \mathsf{E}[Z_1],\; t \cdot \mathrm{Var}(Z_1)).$$

This result follows directly from the central limit theorem for i.i.d. random variables.

Random walks are processes with *independent increments*. That means, the increment $Z_{t+1}$ of the process from time $t$ to time $t+1$ is independent of the past values $X_0, \ldots, X_t$ up to time $t$. In general, it holds for any $s > 0$ that the increment of the process from time $t$ to time $t + s$

$$X_{t+s} - X_t = Z_{t+1} + \ldots + Z_{t+s}$$

is independent of $X_0, \ldots, X_t$. It follows that the best prediction, in terms of mean squared error, for $X_{t+1}$ given $X_0, \ldots, X_t$ is just $X_t + \mathsf{E}[Z_{t+1}]$. As long as the price of only one stock is considered, this prediction rule works quite well. As already as one hundred years ago, Bachelier postulated (assuming $\mathsf{E}[Z_k] = 0$ for all $k$):"The best prediction for the stock price of tomorrow is the price of today."

Processes with independent increments are also *Markov–processes*. In other words, the future evolution of the process in time $t$ depends exclusively on $X_t$, and the value of $X_t$ is independent of the past values $X_0, \ldots, X_{t-1}$. If the increments $Z_k$ and the starting value $X_0$, and hence all $X_t$, can take a finite or countably infinite number of values, then the *Markov–property* is formally

expressed by:

$$\begin{aligned}&\mathrm{P}(a_{t+1} < X_{t+1} < b_{t+1} | X_t = c,\, a_{t-1} < X_{t-1} < b_{t-1}, \ldots,\, a_0 < X_0 < b_0)\\ &\qquad = \mathrm{P}(a_{t+1} < X_{t+1} < b_{t+1} | X_t = c)\,.\end{aligned}$$

If $X_t = c$ is known, additional information about $X_0, \ldots, X_{t-1}$ does not influence the opinion about the range in which $X_t$ will probably fall.

## 4.4 Geometric Random Walks

The essential idea underlying the random walk for real processes is the assumption of mutually independent increments of the order of magnitude for each point in time. However, economic time series in particular do not satisfy the latter assumption. Seasonal fluctuations of monthly sales figures for example are in *absolute terms* significantly greater if the yearly average sales figure is high. By contrast, the relative or percentage changes are stable over time and do not depend on the current level of $X_t$. Analogously to the random walk with i.i.d. absolute increments $Z_t = X_t - X_{t-1}$, a *geometric random walk* $\{X_t;\, t \geq 0\}$ is assumed to have i.i.d. relative increments

$$R_t = \frac{X_t}{X_{t-1}}\,, \quad t = 1, 2, \ldots\,.$$

For example, a geometric *binomial random walk* is given by

$$X_t = R_t \cdot X_{t-1} = X_0 \cdot \Pi_{k=1}^{t} R_k \tag{4.5}$$

where $X_0, R_1, R_2, \ldots$ are mutually independent and for $u > 1$, $d < 1$ :

$$\mathrm{P}(R_k = u) = p\ ,\ \mathrm{P}(R_k = d) = 1 - p\ .$$

Given the independence assumption and $\mathsf{E}[R_k] = (u-d)p + d$ it follows from equation (4.5) that $\mathsf{E}[X_t]$ increases or decreases exponentially as the case may be $\mathsf{E}[R_k] > 1$ or $\mathsf{E}[R_k] < 1$:

$$\mathsf{E}[X_t] = \mathsf{E}[X_0] \cdot (\mathsf{E}[R_1])^t = \mathsf{E}[X_0] \cdot \{(u-d)p + d\}^t.$$

If $\mathsf{E}[R_k] = 1$ the process is on average stable, which is the case for

$$p = \frac{1-d}{u-d}\,.$$

For a recombining process, i.e. $d = \frac{1}{u}$, this relationship simplifies to

$$p = \frac{1}{u+1}\,.$$

Taking logarithms in equation (4.5) yields:

$$\log X_t = \log X_0 + \sum_{k=1}^{t} \log R_k \ .$$

The process $\tilde{X}_t = \log X_t$ is itself an ordinary binomial process with starting value $\log X_0$ and increments $Z_k = \log R_k$:

$$\mathrm{P}(Z_k = \log u) = p, \quad \mathrm{P}(Z_k = \log d) = 1 - p \ .$$

For $t$ large, $\tilde{X}_t$ is approximately normally distributed, i.e. $X_t = \exp(\tilde{X}_t)$ is approximately lognormally distributed.

## 4.5 Binomial Models with State Dependent Increments

Binomial processes and more general random walks model the stock price at best locally. They proceed from the assumption that the distribution of the increments $Z_t = X_t - X_{t-1}$ are the same for each value $X_t$, regardless of whether the stock price is substantially greater or smaller than $X_0$. Absolute increments $X_t - X_{t-1} = (R_t - 1)\, X_{t-1}$ of a geometric random walk depend on the level of $X_{t-1}$. Thus, geometric random walks are processes which do not have independent absolute increments. Unfortunately, when modelling the stock price dynamics globally, the latter processes are too simple to explain the impact of the current price level on the future stock price evolution. A class of processes which take this effect into account are binomial processes with state dependent (and possibly time dependent) increments:

$$X_t = X_{t-1} + Z_t \ , \quad t = 1, 2, \ldots \tag{4.6}$$

$$\mathrm{P}(Z_t = u) = p(X_{t-1}, t) \, , \ \ \mathrm{P}(Z_t = -d) = 1 - p(X_{t-1}, t) \, .$$

Since the distribution of $Z_t$ depends on the state $X_{t-1}$ and possibly on time $t$, increments are neither independent nor identically distributed. The deterministic functions $p(x, t)$ associate a probability to each possible value of the process at time $t$ and to each $t$. Stochastic processes $\{X_t;\ t \geq 0\}$ which are constructed as in (4.6) are still *markovian* but without having independent increments.

Accordingly, geometric binomial processes with state dependent relative increments can be defined (for $u > 1,\ d < 1$):

$$X_t = R_t \cdot X_{t-1} \tag{4.7}$$

$$\mathrm{P}(R_t = u) = p(X_{t-1}, t)\,, \;\; \mathrm{P}(R_t = d) = 1 - p(X_{t-1}, t)\,.$$

Processes as defined in (4.6) and (4.7) are mainly of theoretic interest, since without further assumptions it is rather difficult to estimate the probabilities $p(x, t)$ from observed stock prices. But generalized binomial models (as well as the trinomial models) can be used to solve differential equations numerically, as the Black–Scholes equation for American options for example.

## 4.6 Recommended Literature

Numerous textbooks deal with stochastic processes in discrete time. We recommend a basic book by Brzezniak and Zastawniak (1999). The classic reference work is, of course, Gikhman and Skorokhod (1974). There are 3 volumes that are reprinted for today's applications.

# 5 Stochastic Integrals and Differential Equations

This chapter provides the tools needed for option pricing. The field of stochastic processes in continuous time, which are defined as solutions of stochastic differential equations, has an important role to play. To illustrate these notions we use repeated approximations by stochastic processes in discrete time and refer to the results from Chapter 4.

A stochastic process in continuous time $\{X_t;\ t \geq 0\}$ consists of chronologically ordered random variables, but here the variable $t$ is continuous, i.e. $t$ is a positive real number.

Stock prices are actually processes in discrete time. But to derive the Black-Scholes equation they are approximated by continuous time processes which are easier to handle analytically. However the simulation on a computer of such processes or the numerical computation of say American options, is carried out by means of discrete time approximations. We therefore switch the time scale between discrete and continuous depending on what is more convenient for the actual computation.

## 5.1 Wiener Process

We begin with a simple symmetric random walk $\{X_n;\ n \geq 0\}$ starting in 0 ($X_0 = 0$). The increments $Z_n = X_n - X_{n-1}$ are i.i.d. with :

$$\mathrm{P}(Z_n = 1) = \mathrm{P}(Z_n = -1) = \frac{1}{2}\,.$$

By shortening the period of time of two successive observations we accelerate the process. Simultaneously, the increments of the process become smaller during the shorter period of time. More precisely, we consider a stochastic process $\{X_t^{\Delta};\ t \geq 0\}$ in continuous time which increases or decreases in a time step $\Delta t$ with probability $\frac{1}{2}$ by $\Delta x$. Between these jumps the process is constant (alternatively we could interpolate linearly). At time $t = n \cdot \Delta t$ the

process is:

$$X_t^\Delta = \sum_{k=1}^{n} Z_k \cdot \Delta x = X_n \cdot \Delta x$$

where the increments $Z_1\Delta x, Z_2\Delta x, \ldots$ are mutually independent and take the values $\Delta x$ or $-\Delta x$ with probability $\frac{1}{2}$ respectively. From Section 4.1 we know:

$$\mathsf{E}[X_t^\Delta] = 0\ , \qquad \mathrm{Var}(X_t^\Delta) = (\Delta x)^2 \cdot \mathrm{Var}(X_n) = (\Delta x)^2 \cdot n = t \cdot \frac{(\Delta x)^2}{\Delta t}\ .$$

Now, we let $\Delta t$ and $\Delta x$ become smaller. For the process in the limit to exist in a reasonable sense, $\mathrm{Var}(X_t^\Delta)$ must be finite. On the other hand, $\mathrm{Var}(X_t^\Delta)$ should not converge to 0, since the process would no longer be random. Hence, we must choose:

$$\Delta t \to 0,\ \Delta x = c \cdot \sqrt{\Delta t}\,,\ \text{such that}\ \ \mathrm{Var}(X_t^\Delta) \to c^2 t\,.$$

If $\Delta t$ is small, then $n = t/\Delta t$ is large. Thus, the random variable $X_n$ of the ordinary symmetric random walk is approximately $\mathrm{N}(0, n)$ distributed, and therefore for all $t$ (not only for $t$ such that $t = n\ \Delta t$):

$$\mathcal{L}(X_t^\Delta) \approx \mathrm{N}(0,\ n(\Delta x)^2) \approx \mathrm{N}(0,\ c^2 t)\ .$$

Thus the limiting process $\{X_t;\ t \geq 0\}$ which we obtain from $\{X_t^\Delta;\ t \geq 0\}$ for $\Delta t \to 0,\ \Delta x = c\ \sqrt{\Delta t}$ has the following properties:

(i) $X_t$ is $\mathrm{N}(0, c^2 t)$ distributed for all $t \geq 0$.

(ii) $\{X_t;\ t \geq 0\}$ has *independent increments*, i.e. for $0 \leq s < t$, $X_t - X_s$ is independent of $X_s$ (since the random walk $\{X_n;\ n \geq 0\}$ defining $\{X_t^\Delta;\ t \geq 0\}$ has independent increments).

(iii) For $0 \leq s < t$ the increment $(X_t - X_s)$ is $\mathrm{N}(0,\ c^2 \cdot (t-s))$ distributed, i.e. its distribution depends exclusively on the length $t - s$ of the time interval in which the increment is observed (this follows from (i) and (ii) and the properties of the normal distribution).

A stochastic process $\{X_t;\ t \geq 0\}$ in continuous time satisfying (i)–(iii) is called *Wiener process* or *Brownian motion* starting in 0 ($X_0 = 0$). The standard Wiener process resulting from $c = 1$ will be denoted by $\{W_t;\ t \geq 0\}$. For this process it holds for all $0 \leq s < t$

$$\mathsf{E}[W_t] = 0,\ \mathrm{Var}(W_t) = t$$

$$\begin{aligned}\mathrm{Cov}(W_t, W_s) &= \mathrm{Cov}(W_t - W_s + W_s, W_s)\\ &= \mathrm{Cov}(W_t - W_s, W_s) + \mathrm{Cov}(W_s, W_s)\\ &= 0 + \mathrm{Var}(W_s) = s\end{aligned}$$

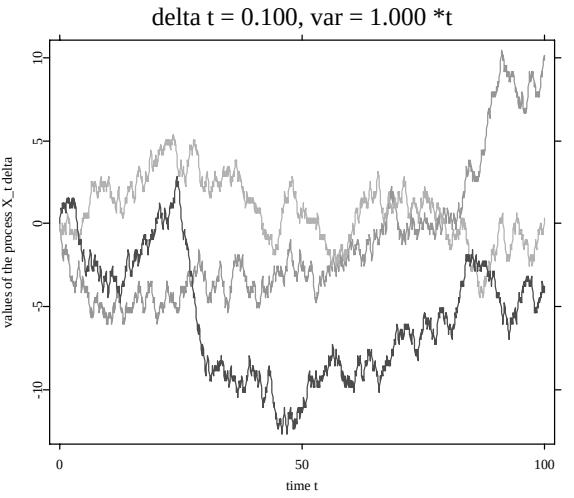



Figure 5.1: Typical paths of a Wiener process. SFEWienerProcess

As for every stochastic process in continuous time, we can consider a path or realisation of the Wiener process as a *randomly chosen function* of time. With some major mathematical instruments it is possible to show that the paths of a Wiener process are continuous with probability 1:

$$\mathrm{P}(W_t \text{ is continuous as a function of } t) = 1\,.$$

That is to say, the Wiener process has no jumps. But $W_t$ fluctuates heavily: the paths are continuous but highly erratic. In fact, it is possible to show that the paths are not differentiable with probability 1.

Being a process with independent increments the Wiener process is *markovian*. For $0 \le s < t$ it holds $W_t = W_s + (W_t - W_s)$, i.e. $W_t$ depends only on $W_s$ and on the increment from time $s$ to time $t$:

$$\begin{aligned} & \mathrm{P}(a < W_t < b \,|W_s = x \text{ , information about } W_{t'},\ 0 \le t' < s) \\ = \ & \mathrm{P}(a < W_t < b \,|W_s = x \ ) \end{aligned}$$

A graphical representation of the above equation is given in Figure 5.2.

Using properties (i)–(iii), the distribution of $W_t$ given the outcome $W_s = x$ can be formulated explicitly. Since the increment $(W_t - W_s)$ is $\mathrm{N}(0, t-s)$

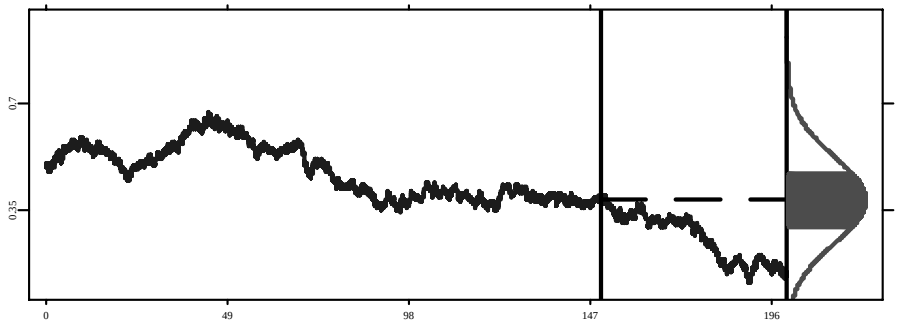



Figure 5.2: Graphical representation of $P(a < W_t < b \,|W_s = w)$.
SFEwienerdens

distributed, $W_t$ is $\mathrm{N}(x, t-s)$ distributed given $W_s = x$ :

$$\mathrm{P}(a < W_t < b \,|W_s = x) = \int_a^b \frac{1}{\sqrt{t-s}} \varphi\big(\frac{y-x}{\sqrt{t-s}}\big) dy.$$

Proceeding from the assumption of a random walk $\{X_n;\, n \geq 0\}$ with drift $\mathsf{E}[X_n] = n(2p-1)$ instead of a symmetric random walk results in a process $X_t^\Delta$ which is no more zero on average, but

$$\begin{aligned} \mathsf{E}[X_t^\Delta] &= n \cdot (2p-1) \cdot \Delta x = (2p-1) \cdot t \, \frac{\Delta x}{\Delta t} \\ \mathrm{Var}(X_t^\Delta) &= n \; 4p(1-p) \cdot (\Delta x)^2 = 4p(1-p) \cdot t \cdot \frac{(\Delta x)^2}{\Delta t}. \end{aligned}$$

For $\Delta t \to 0$, $\Delta x = \sqrt{\Delta t}$, $p = \frac{1}{2}(1 + \mu\sqrt{\Delta t})$ we obtain for all $t$:

$$\mathsf{E}[X_t^\Delta] \to \mu t\,, \;\; \mathrm{Var}(X_t^\Delta) \to t.$$

The limiting process is a Wiener process $\{X_t;\; t \geq 0\}$ with *drift* or *trend* $\mu t$. It results from the standard Wiener process:

$$X_t = \mu t + W_t.$$

Hence, it behaves in the same way as the standard Wiener process but it fluctuates on average around $\mu$ instead of 0. If $(\mu > 0)$ the process is increasing linearly on average, and if $(\mu < 0)$ it is decreasing linearly on average.

## 5.2 Stochastic Integration

In order to introduce a stochastic process as a solution of a stochastic differential equation we introduce the concept of the Itô–integral: a stochastic integral with respect to a Wiener process. Formally the construction of the Itô–integral is similar to the Stieltjes–integral. However, instead of integrating with respect to a deterministic function (Stieltjes–integral), the Itô–integral integrates with respect to a random function, more precisely, the path of a Wiener process. Since the integrant itself can be random, i.e. it can be a path of a stochastic process, one has to analyze the mutual dependencies of the integrant and the Wiener process.

Let $\{Y_t;\ t \geq 0\}$ be the process to integrate and let $\{W_t;\ t \geq 0\}$ be a standard Wiener process. The definition of a stochastic integral assumes that $\{Y_t;\ t \geq 0\}$ is non-anticipating. Intuitively, it means that the process up to time $s$ does not contain any information about future increments $W_t - W_s$ , $t > s$ , of the Wiener process. In particular, $Y_s$ is independent of $W_t - W_s$ .

An integral of a function is usually defined as the limit of the sum of the suitably weighted function. Similarly, the *Itô integral* with respect to a Wiener process is defined as the limit of the sum of the (randomly) weighted (random) function $\{Y_t;\ t \geq 0\}$ :

$$I_n = \sum_{k=1}^{n} Y_{(k-1)\Delta t} \cdot (W_{k\Delta t} - W_{(k-1)\Delta t}) \ , \quad \Delta t = \frac{t}{n} \tag{5.1}$$

$$\int_0^t Y_s \, dW_s = \lim_{n\to\infty} I_n \ ,$$

where the limit is to be understood as the limit of a random variable in terms of mean squared error, i.e. it holds

$$\mathsf{E}\{\big[\int_0^t Y_s \, dW_s - I_n\big]^2\} \to 0, \quad n \to \infty.$$

It is important to note, that each summand of $I_n$ is a product of two independent random variables. More precisely, $Y_{(k-1)\Delta t}$, the process to integrate at the left border of the small interval $[(k-1)\Delta t,\ k\Delta t]$ is independent of the increment $W_{k\Delta t} - W_{(k-1)\Delta t}$ of the Wiener process in this interval.

It is not hard to be more precise on the non-anticipating property of $\{Y_t;\ t \geq 0\}$.

**Definition 5.1 (Information structure, non–anticipating)**
*For each $t \geq 0$, $\mathcal{F}_t$ denotes a family of events (having the structure of a $\sigma$–algebra, i.e. certain combinations of events contained in $\mathcal{F}_t$ are again in $\mathcal{F}_t$) which contain the available information up to time $t$. $\mathcal{F}_t$ consists of events from which is known up to time $t$ whether they occurred or not. We assume:*

$$\mathcal{F}_s \subset \mathcal{F}_t \quad \text{for } s < t \quad \text{(information grows as time evolves)}$$

$$\{a < Y_t < b\} \in \mathcal{F}_t \quad \text{($Y_t$ contains no information about events occurring after time $t$)}$$

$$\{a < W_t < b\} \in \mathcal{F}_t$$

$$W_t - W_s \text{ independent of } \mathcal{F}_s \quad \text{for } s < t \quad \text{(the Wiener process is adapted to the evolution of information)}$$

*Then, we call $\mathcal{F}_t$ the information structure at time $t$ and the process $\{Y_t;\ t \geq 0\}$ non–anticipating with respect to the information structure $\mathcal{F}_t;\ t \geq 0$.*

The process $\{Y_t\}$ is called non–anticipating since due to the second assumption it does not anticipate any future information. The evolving information structure $\mathcal{F}_t$ and the random variables $Y_t, W_t$ are adapted to each other.

The integral depends crucially on the point of the interval $[(k-1)\Delta t,\ k\Delta t]$ at which the random variable $Y_s$ is evaluated in (5.1). Consider the example $Y_t = W_t,\ t \geq 0$, i.e. we integrate the Wiener process with respect to itself. As a gedankenexperiment (though experiment) we replace in (5.1) $(k-1)\Delta t$ by an arbitrary point $t(n,k)$ of the interval $[(k-1)\Delta t,\ k\Delta t]$. If we defined:

$$\int_0^t W_s\, dW_s = \lim_{n\to\infty} \sum_{k=1}^{n} W_{t(n,k)}\,(W_{k\Delta t} - W_{(k-1)\Delta t})$$

the expected values would converge as well. Hence by interchanging the sum with the covariance operator we get:

$$\mathsf{E}\quad [\sum_{k=1}^{n} W_{t(n,k)}\,(W_{k\Delta t} - W_{(k-1)\Delta t})] = \sum_{k=1}^{n} \mathrm{Cov}(W_{t(n,k)},\, W_{k\Delta t} - W_{(k-1)\Delta t})$$

$$= \quad \sum_{k=1}^{n} \{t(n,k) - (k-1)\Delta t\} \to \mathsf{E}[\int_0^t W_s\, dW_s]\,.$$

For $t(n,k) = (k-1)\Delta t$ – which is the case for the Itô–integral – we obtain 0, for $t(n,k) = k\Delta t$ we obtain $n \cdot \Delta t = t$, and for suitably chosen sequences $t(n,k)$ we could obtain, for the expectation of the stochastic integral, any value between 0 and $t$. In order to assign a unique value to $\int_0^t W_s\ dW_s$, we have to agree on a certain sequence $t(n,k)$.

To illustrate how Itô–integrals are computed, and also that other than the usual computation rules have to be applied, we show that:

$$\int_0^t W_s \; dW_s = \frac{1}{2}(W_t^2 - W_0^2) \; - \; \frac{t}{2} = \frac{1}{2}(W_t^2 - t) \tag{5.2}$$

Summing the differences $W_{k\Delta t}^2 - W_{(k-1)\Delta t}^2$, all terms but the first and the last cancel out and remembering that $n\Delta t = t$ we get

$$\begin{aligned}
\frac{1}{2}(W_t^2 - W_0^2) &= \frac{1}{2}\sum_{k=1}^{n}(W_{k\Delta t}^2 - W_{(k-1)\Delta t}^2) \\
&= \frac{1}{2}\sum_{k=1}^{n}(W_{k\Delta t} - W_{(k-1)\Delta t})(W_{k\Delta t} + W_{(k-1)\Delta t}) \\
&= \frac{1}{2}\sum_{k=1}^{n}(W_{k\Delta t} - W_{(k-1)\Delta t})^2 \\
&\quad + \sum_{k=1}^{n}(W_{k\Delta t} - W_{(k-1)\Delta t}) \; W_{(k-1)\Delta t} \, .
\end{aligned}$$

While the second term converges to $\int_0^t W_s \; dW_s$ , the first term is a sum of $n$ independent identically distributed random variables and which is thus approximated due to the law of large numbers by its expected value

$$\frac{n}{2}\,\mathsf{E}[(W_{k\Delta t} - W_{k-1)\Delta t})^2] = \frac{n}{2}\Delta t = \frac{t}{2} \, .$$

For smooth functions $f_s$, for example continuously differentiable functions, it holds $\int_0^t f_s \; df_s = \frac{1}{2}(f_t^2 - f_0^2)$. However, the stochastic integral (5.2) contains the additional term $-\frac{t}{2}$ since the local increment of the Wiener process over an interval of length $\Delta t$ is of the size of its standard deviation – that is $\sqrt{\Delta t}$. The increment of a smooth function $f_s$ is proportional to $\Delta t$, and therefore considerably smaller than the increment of the Wiener process for $\Delta t \to 0$.

## 5.3 Stochastic Differential Equations

Since the Wiener process fluctuates around its expectation 0 it can be approximated by means of symmetric random walks. As for random walks we are interested in stochastic processes in continuous time which grow on average, i.e. which have a *trend* or *drift*. Proceeding from a Wiener process with arbitrary $\sigma$ (see Section 5.1) we obtain the generalized Wiener process

$\{X_t;\ t \geq 0\}$ with *drift rate* $\mu$ and variance $\sigma^2$ :

$$X_t = \mu \cdot t + \sigma \cdot W_t \quad , \quad t \geq 0\,. \tag{5.3}$$

The general Wiener process $X_t$ is at time $t$, $\mathrm{N}(\mu t,\, \sigma^2 t)$–distributed. For its increment in a small time interval $\Delta t$ we obtain

$$X_{t+\Delta t} - X_t = \mu \cdot \Delta t + \sigma(W_{t+\Delta t} - W_t)\,.$$

For $\Delta t \to 0$ use the differential notation:

$$dX_t = \mu \cdot dt + \sigma \cdot dW_t \tag{5.4}$$

This is only a different expression for the relationship (5.3) which we can also write in integral form:

$$X_t = \int_0^t \mu\, ds + \int_0^t \sigma dW_s \tag{5.5}$$

Note, that from the definition of the stochastic integral it follows directly that $\int_0^t d\ W_s = W_t - W_0 = W_t$ .

The differential notation (5.4) proceeds from the assumption that both the local drift rate given by $\mu$ and the local variance given by $\sigma^2$ are constant. A considerably larger class of stochastic processes which is more suited to model numerous economic and natural processes is obtained if $\mu$ and $\sigma^2$ in (5.4) are allowed to be time and state dependent. Such processes $\{X_t;\ t \geq 0\}$ , which we call Itô–processes, are defined as solutions of *stochastic differential equations*:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t \tag{5.6}$$

Intuitively, this means:

$$X_{t+\Delta t} - X_t = \mu(X_t, t)\Delta t + \sigma(X_t, t)(W_{t+\Delta t} - W_t),$$

i.e. the process' increment in a small interval of length $\Delta t$ after time $t$ is $\mu(X_t, t)\cdot\Delta t$ plus a random fluctuation which is $\mathrm{N}(0,\ \sigma^2(X_t, t)\cdot\Delta t)$ distributed. A precise definition of a solution of (5.6) is a stochastic process fulfilling the integral equation

$$X_t - X_0 = \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_s \tag{5.7}$$

In this sense (5.6) is only an abbreviation of (5.7). For $0 \leq t' < t$ , it follows immediately:

$$X_t = X_{t'} + \int_{t'}^t \mu(X_s, s)ds + \int_{t'}^t \sigma(X_s, s)dW_s\,.$$

Since the increment of the Wiener process between $t'$ and $t$ does not dependent on the events which occurred up to time $t'$, it follows that an Itô–process is *Markovian.*

Discrete approximations of (5.6) and (5.7) which can be used to simulate Itô–processes are obtained by observing the process between 0 and $t$ only at evenly spaced points in time $k\Delta t$, $k = 0, \ldots, n$ , $n\Delta t = t$ .

With $X_k = X_{k\Delta t}$ and $Z_k = (W_{k\Delta t} - W_{(k-1)\Delta t})/\sqrt{\Delta t}$ we get

$$X_{k+1} - X_k = \mu(X_k, k) \cdot \Delta t + \sigma(X_k, k) \cdot Z_{k+1} \cdot \sqrt{\Delta t}$$

or rather with the abbreviations $\mu_k(X) = \mu(X, k)\Delta t$ , $\sigma_k(X) = \sigma(X, k)\sqrt{\Delta t}$ :

$$X_n - X_0 = \sum_{k=1}^{n} \mu_{k-1}(X_{k-1}) + \sum_{k=1}^{n} \sigma_{k-1}(X_{k-1}) \cdot Z_k$$

with identical independently distributed N(0, 1)–random variables $Z_1, Z_2, \ldots$ .

**Example 5.1**
*The Ornstein-Uhlenbeck (OU) process, a mean-reverting process, represents a well-known example of a process with a constant variance $\sigma(X_t, t) = \sigma$ and nonconstant drift $\mu(X_t, t) = \alpha(\mu - X_t)$. The OU process has the following form:*

$$dX_t = \alpha(\mu - X_t)dt + \sigma dW_t, \tag{5.8}$$

*where $\alpha$, $\mu$ and $\sigma$ are positive constants. The drift term $\alpha(\mu - X_t)$ in (5.8) is negative when $X_t > \mu$ and it is positive when $X_t < \mu$; so, even if the process will never be free of random fluctuations, we can expect $X_t$ to revert back to $\mu$ whenever it has drifted away. In addition, because of the constant local variance $\sigma^2$ (not proportional to the level as in the case of geometric Brownian motion), we expect $X_t$ to fluctuate vigorously and to make many crossings of the $\mu$-level. The OU process can be approximated on a discrete grid $t_i = i\Delta t$, $i = 0, 1, \ldots$ by*

$$X_{t_i} - X_{t_{i-1}} = \alpha\mu\Delta t - \alpha X_{t_{i-1}}\Delta t + \sigma\sqrt{\Delta t}Z_i$$

*where $(Z_i)$ are independent standard normal random variables. Hence, the OU process is the limit of AR(1) processes $X_i = c + \phi X_{i-1} + \varepsilon_i$.*

**Example 5.2**
*The Cox-Ingersoll-Ross (CIR) process, also referred to as s square-root diffusion process, represents an example of a stochastic process where the variance*

*as well as the drift term are not constant. The CIR Process is given by*

$$dX_t = \alpha(\mu - X_t)dt + \sigma\sqrt{X_t}dW_t, \tag{5.9}$$

*with a drift term* $\mu(X_t, t) = \alpha(\mu - X_t)$ *and a local variance* $\sigma(X_t, t) = \sigma\sqrt{X_t}$*;* $\alpha$*,* $\mu$*,* $\sigma$ *are positive constants. Figure 5.3 shows a typical path of a CIR process.*

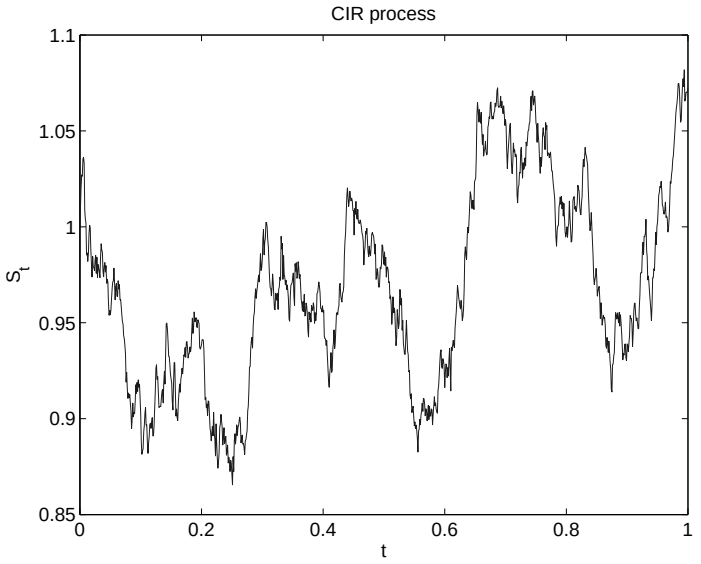



Figure 5.3: Typical path of a CIR process ($\alpha = 5$, $\mu = 1$, $\sigma = 0.2$).
SFESimCIR

## 5.4 The Stock Price as a Stochastic Process

Stock prices are stochastic processes in *discrete time* which only take *discrete values* due to the limited measurement scale. Nevertheless, stochastic processes in *continuous time* are used as models since they are analytically easier to handle than discrete models, e.g. the binomial or trinomial process. However, the latter are more intuitive and prove to be very useful in simulations.

Two features of the general Wiener process $dX_t = \mu dt + \sigma\, dW_t$ make it an unsuitable model for stock prices. First, it allows for negative stock prices,

and second the local variability is higher for high stock prices. Hence, stock prices $S_t$ are modeled by means of the more general Itô–process:

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t\,.$$

This model, however, does depend on the unknown functions $\mu(X, t)$ and $\sigma(X, t)$. A useful and simpler variant utilizing only two unknown real model parameters $\mu$ and $\sigma$ can be justified by the following reflection: The percentage return on the invested capital should on average not depend on the stock price at which the investment is made, and of course, should not depend on the *currency unit* (EUR, USD, ...) in which the stock price is quoted. Furthermore, the average return should be proportional to the investment horizon, as it is the case for other investment instruments. Putting things together, we request:

$$\frac{\mathsf{E}[dS_t]}{S_t} \;=\; \frac{\mathsf{E}[S_{t+dt} - S_t]}{S_t} \;=\; \mu \cdot dt\,.$$

Since $\mathsf{E}[dW_t] = 0$ this condition is satisfied if

$$\mu(S_t, t) = \mu \cdot S_t\,,$$

for given $S_t$. Additionally,

$$\sigma(S_t, t) = \sigma \cdot S_t$$

takes into consideration that the absolute size of the stock price fluctuation is proportional to the currency unit in which the stock price is quoted. In summary, we model the stock price $S_t$ as a solution of the stochastic differential equation

$$dS_t = \mu \cdot S_t\, dt + \sigma \cdot S_t \cdot dW_t\,,$$

where $\mu$ is the *expected return* on the stock, and $\sigma$ the *volatility*. Such a process is called *geometric Brownian motion* because

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t\,.$$

We can interpret this equation as follows. For small time $\Delta t$ the change in the stock can be approximated by

$$S_{t+\Delta t} - S_t = \mu S_t \Delta t + \sigma S_t (W_{t+\Delta t} - W_t),$$

where $W_{t+\Delta t} - W_t$ represents the change of the Brownian motion over the time interval $\Delta t$. As $\Delta t$ becomes smaller the approximation becomes more accurate. A typical path of a geometric Brownian motion is plotted in Figure

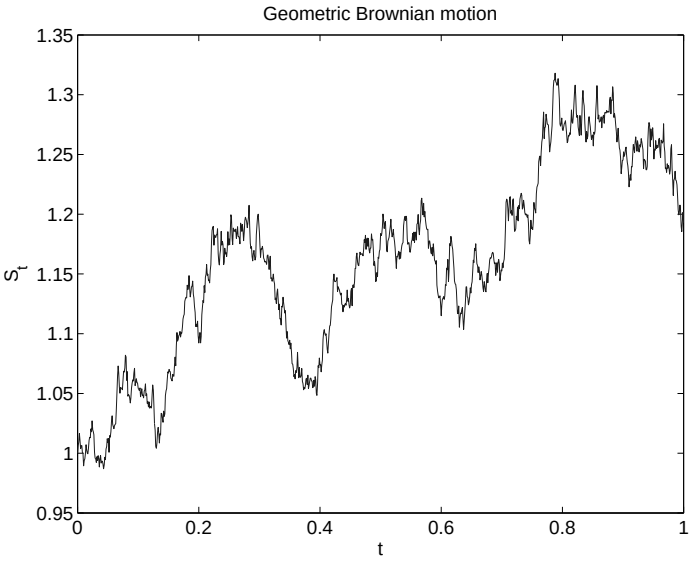



Figure 5.4: Typical path of a geometric Brownian motion ($\mu = 0.05$, $\sigma = 0.2$).
SFEGBMProcess

5.4. The rough nature of the path corresponds to the fact that no paths of a (geometric) Brownian motion are differentiable anywhere. We have chosen as parameters a mean return of 5% and a volatility of 20% which can be regarded as realistic values for stock price processes.

By applying Itôs lemma, which we introduce in Section 5.5, it can be shown that for a suitable Wiener process $\{Y_t;\ t \geq 0\}$ it holds

$$S_t = e^{Y_t} \quad \text{bzw.} \quad Y_t = \log S_t\,.$$

Since $Y_t$ is normally distributed, $S_t$ is lognormally distributed. As random walks can be used to approximate the general Wiener process, geometric random walks can be used to approximate geometric Brownian motion and thus this simple model for the stock price.

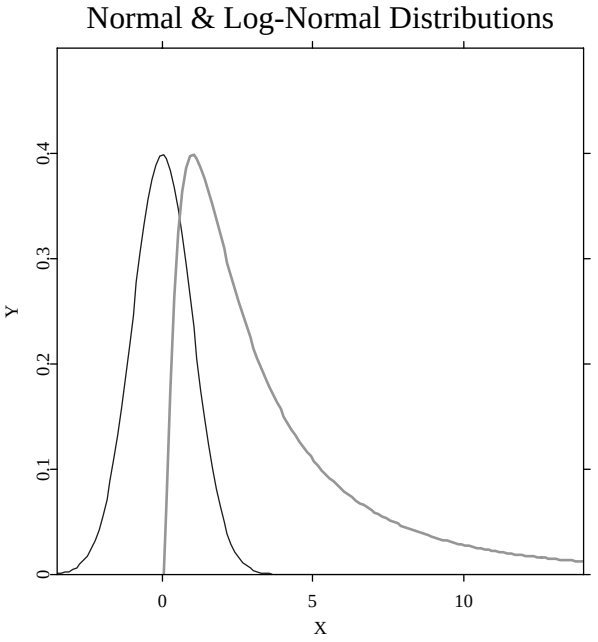



Figure 5.5: Density comparison of lognormally and normally distributed random variables. SFELogNormal

## 5.5 Itô's Lemma

A crucial tool in dealing with stochastic differential equations is Itôs lemma. If $\{X_t,\ t \geq 0\}$ is an Itô–process:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)\, dW_t\,, \tag{5.10}$$

one is often interested in the dynamics of stochastic processes which are functions of $X_t$: $Y_t = g(X_t)$. Then $\{Y_t;\ t \geq 0\}$ can also be described by a solution of a stochastic differential equation from which interesting properties of $Y_t$ can be derived, as for example the average growth in time $t$.

For a heuristic derivation of the equation for $\{Y_t;\ t \geq 0\}$ we assume that $g$ is differentiable as many times as necessary. From a Taylor expansion it follows

that:

$$\begin{aligned} Y_{t+dt} - Y_t &= g(X_{t+dt}) - g(X_t) \\ &= g(X_t + dX_t) - g(X_t) \\ &= \frac{dg}{dX}(X_t) \cdot dX_t + \frac{1}{2}\frac{d^2g}{dX^2}(X_t) \cdot (dX_t)^2 + \dots \end{aligned} \tag{5.11}$$

where the dots indicate the terms which can be neglected (for $dt \to 0$). Due to equation (5.10) the drift term $\mu(X_t, t)dt$ and the volatility term $\sigma(X_t, t)dW_t$ are the dominant terms since for $dt \to 0$ they are of size $dt$ and $\sqrt{dt}$ respectively.

In doing this, we use the fact that $\mathsf{E}[(dW_t)^2] = dt$ and $dW_t = W_{t+dt} - W_t$ is of the size of its standard deviation, $\sqrt{dt}$. We neglect terms which are of a smaller size than $dt$. Thus, we can express $(dX_t)^2$ by a simpler term:

$$\begin{aligned} (dX_t)^2 &= (\mu(X_t, t)dt + \sigma(X_t, t)\, dW_t)^2 \\ &= \mu^2(X_t, t)(dt)^2 + 2\mu(X_t, t)\ \sigma(X_t, t)\ dt\ dW_t + \sigma^2(X_t, t)(dW_t)^2\,. \end{aligned}$$

We see that the first and the second terms are of size $(dt)^2$ and $dt \cdot \sqrt{dt}$ respectively. Therefore, both can be neglected. However, the third term is of size $dt$. More precisely, it can be shown that $dt \to 0$ :

$$(dW_t)^2 = dt.$$

Thanks to this identity, calculus rules for stochastic integrals can be derived from the rules for deterministic functions (as Taylor expansions for example). Neglecting terms which are of smaller size than $dt$ we obtain the following version of *Itôs lemma* from (5.11):

**Lemma 5.1 (Itôs Lemma)**

$$\begin{aligned} dY_t &= dg(X_t) \\ &= \left(\frac{dg}{dX}(X_t) \cdot \mu(X_t, t) + \frac{1}{2}\frac{d^2g}{dX^2}(X_t) \cdot \sigma^2(X_t, t)\right) dt \\ &\quad + \frac{dg}{dX}(X_t) \cdot \sigma(X_t, t)\, dW_t \end{aligned}$$

*or - dropping the time index $t$ and the argument $X_t$ of the function $g$ and its derivatives:*

$$dg = \left(\frac{dg}{dX}\mu(X, t) + \frac{1}{2}\frac{d^2g}{dX^2}\sigma^2(X, t)\right) dt + \frac{dg}{dX}\sigma(X, t)dW_t\,.$$

**Example 5.3**
*Consider* $Y_t = \log S_t$ *the logarithm of the geometric Brownian motion. For* $g(X) = \log X$ *we obtain* $\frac{dg}{dX} = \frac{1}{X}$ , $\frac{d^2g}{dX^2} = -\frac{1}{X^2}$ . *Applying Itôs lemma for the geometric Brownian motion with* $\mu(X,t) = \mu X$ , $\sigma(X,t) = \sigma\, X$ *we get:*

$$\begin{aligned} dY_t &= \big(\frac{1}{S_t}\mu\, S_t - \frac{1}{2}\,\frac{1}{S_t^2}\,\sigma^2\, S_t^2\big)dt + \frac{1}{S_t}\cdot\sigma\, S_t\, dW_t \\ &= (\mu - \frac{1}{2}\,\sigma^2)dt + \sigma\, dW_t \end{aligned}$$

*The* logarithm of the stock price *is a generalised Wiener process with drift rate* $\mu^* = \mu - \frac{1}{2}\,\sigma^2$ *and variance rate* $\sigma^2$. *Since* $Y_t$ *is* $N(\mu^* t,\, \sigma^2 t)$*–distributed* $S_t$ *is itself* lognormally distributed *with parameters* $\mu^* t$ *and* $\sigma^2 t$ .

A generalized version of Itôs lemma for functions $g(X,t)$ which are allowed to depend on time $t$ is:

**Lemma 5.2 (Itôs lemma for functions depending explicitly on time)**

$$dg = \left(\frac{\partial g}{\partial X}\cdot\mu(X,t) + \frac{1}{2}\,\frac{\partial^2 g}{\partial X^2}\,\sigma^2(X,t) + \frac{\partial g}{\partial t}\right)dt + \frac{\partial g}{\partial X}\,\sigma(X,t)dW_t \quad (5.12)$$

$Y_t = g(X_t,t)$ is again an Itô process, but this time the drift rate is augmented by an additional term $\frac{\partial g}{\partial t}(X_t,t)$.

**Example 5.4**
*Consider a forward contract on a non-dividend paying stock. Let* $S_0$ *denotes spot price at time* 0 *and* $T$ *is the time to maturity of the forward contract. We assume that the risk-free interest rate* $r$ *is constant for all maturities. The forward price at time* 0 *is then given by*

$$F_0 = S_0 e^{rT} \quad (5.13)$$

*and at a general time* $t < T$ *by*

$$F_t = S_t e^{r(T-t)}. \quad (5.14)$$

*Assuming that the process* $\{S_t, t \geq 0\}$ *is given by (5.10), we use Itô's lemma to determine the process for* $F_t$. *For* $g(X,t) = Xe^{r(T-t)}$ *we obtain:*

$$\frac{dg}{dX} = e^{r(T-t)} \qquad \frac{d^2g}{dX^2} = 0 \qquad \frac{dg}{dt} = -rXe^{r(T-t)}.$$

*Applying (5.12) to the geometric Brownian motion with* $\mu(X,t) = \mu X$ , $\sigma(X, t) = \sigma X$ *we get*

$$dF_t = \left\{ e^{r(T-t)}\mu S_t - rS_t e^{r(T-t)} \right\} dt + e^{r(T-t)}\sigma S_t dW_t.$$

*Substituting* $F_t = S_t e^{r(T-t)}$*, this becomes*

$$dF_t = (\mu - r)F_t dt + \sigma F_t dW_t.$$

*Like* $S_t$*, the process* $\{F_t, t \geq 0\}$ *follows a geometric Brownian motion, however with an expected growth rate of* $\mu - r$ *rather than* $\mu$*.*

## 5.6 Recommended Literature

This chapter briefly summarises results which belong to the main topics of stochastic analysis. Numerous textbooks, of diverse levels, introduce the calculus of stochastic integrals and stochastic differential equations, see for example, von Weizsäcker and Winkler (1990), Mikosch (1998) or Karatzas and Shreve (1999).

# 6 Black–Scholes Option Pricing Model

## 6.1 Black–Scholes Differential Equation

Simple generally accepted economic assumptions are insufficient to develop a rational option pricing theory. Assuming a perfect financial market in Section 2.1 leads to elementary arbitrage relations which options have to fulfill. While these relations can be used as a verification tool for sophisticated mathematical models, they do not provide an explicit option pricing function depending on parameters such as time and the stock price as well as the options underlying parameters $K$, $T$. To obtain such a pricing function the value of the underlying financial instrument (stock, currency, ...) has to be modelled. In general, the underlying instrument is assumed to follow a stochastic process either in discrete or in continuous time. While the latter are analytically easier to handle, the former, which we will consider as approximations of continuous time processes for the time being, are particularly useful for numerical computations. In the second part of this text, the discrete time version will be discussed as financial time series models.

A model for stock prices which is frequently used and is also the basis of the classical Black–Scholes approach, is the so–called geometric Brownian motion. In this model the stock price $S_t$ is a solution of the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{6.1}$$

Equivalently, the process of stock price returns can be assumed to follow a standard Brownian motion, i.e.

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t. \tag{6.2}$$

The drift $\mu$ is the expected return on the stock in the time interval $dt$. The volatility $\sigma$ is a measure of the return variability around its expectation $\mu$. Both parameters $\mu$ and $\sigma$ are dependent on each other and are important factors of the investors' risk preferences involved in the investment decision:

The higher the expected return $\mu$, the higher, in general, the risk quantified by $\sigma$.

Modelling the underlying as geometric Brownian motion provides a useful approximation to stock prices accepted by practitioners for short and medium maturity. In real practice, numerous model departures are known: in some situations the volatility function $\sigma(x,t)$ of the general model (5.10) is different from the linear specification $\sigma \cdot x$ of geometric Brownian motion. The Black–Scholes' approach is still used to approximate option prices, and its basic idea to derive option prices can be applied to more general stock price models.

Black–Scholes' approach relies on the idea introduced in Chapter 2, i.e. duplicating the portfolio consisting of the option by means of a second portfolio consisting exclusively of financial instruments whose values are known. The duplicating portfolio is chosen in such a way both portfolios have equal values at the option's maturity $T$. It then follows from the assumption of a perfect financial market, and in particular of the lack of arbitrage opportunities, that both portfolios must have equal values at any time prior to time $T$. The duplicating portfolio can be created in two equivalent ways which we illustrate with an example of a call option on a stock with price $S_t$:

1. Consider a portfolio consisting of one call of which the price is to be computed. The duplicating portfolio is composed of stocks and risk–less zero bonds of which the quantity adapts continuously to changes in the stock price. Without loss of generality, the zero bond's nominal value can be set equal to one since the number of zero bonds in the duplicating portfolio can be chosen arbitrarily. At time $t$ the two portfolios consist of:

**Portfolio** $A$**:** One call option (long position) with delivery price $K$ and maturity date $T$.

**Portfolio** $B$**:** $n_t = n(S_t,t)$ stocks and $m_t = m(S_t,t)$ zero bonds with nominal value $B_T = 1$ and maturity date $T$.

2. Consider a perfect hedge–portfolio, which consists of stocks and written calls (by means of short selling). Due to a dynamic hedge–strategy the portfolio bears no risk at any time, i.e. profits due to the calls are neutralized by losses due to the stocks. Correspondingly, the duplicating portfolio is also risk free and consists exclusively of zero bonds. Again, the positions are adjusted continuously to changes in the stock price. At time $t$ the two portfolios are composed of:

**Portfolio** $A$**:** One stock and $n_t = n(S_t,t)$ (by means of short selling) written call options on the stock with delivery price $K$ and maturity date $T$.

**Portfolio** $B$: $m_t = m(S_t, t)$ zero bonds with face value $B_T = 1$ and maturity date $T$.

Let $T^* = T$ be the time when the call option expires as worthless, and otherwise let $T^*$ be the time at which the option is exercised. While for a European call option it holds $T^* = T$ at any time, an American option can be exercised prior to maturity. We will see that both in 1. the call value is equal to the value of the duplicating portfolio, and in 2. the hedge–portfolio's value equals the value of the risk free zero bond portfolio at any time $t \leq T^*$, and thus the same partial differential equation for the call value results; this is called *Black–Scholes equation*.

The Black–Scholes approach can be applied to any financial instrument $\mathcal{U}$ contingent on an underlying with price $S_t$ if the latter price follows a geometric Brownian motion, and if the derivatives price $F_t$ is a function only of the price $S_t$ and time: $F_t = F(S_t, t)$. Then, according to the theorem below, a portfolio duplicating the financial instrument exists, and the approach illustrated in 1. can be applied to price the instrument. Pricing an arbitrary derivative the duplicating portfolio must not only have the same value as the derivative at exercising time $T^*$, but also the same cash flow pattern, i.e. the duplicating portfolio has to generate equal amounts of withdrawal profits or contributing costs as the derivative. The existence of a perfect hedge–portfolio of approach 2. can be shown analogously.

**Theorem 6.1**
*Let the value $S_t$ of an object be a geometric Brownian motion (6.1). Let $\mathcal{U}$ be a derivative contingent on the object and maturing in $T$. Let $T^* \leq T$ be the time at which the derivative is exercised, or $T^* = T$ if it is not. Let the derivative's value at any time $t \leq T^*$ be given by a function $F(S_t, t)$ of the object's price and time.*

*a) A portfolio exists consisting of the underlying object and risk free bonds which duplicates the derivative in the sense that it generates up to time $T^*$ the same cash flow pattern as $\mathcal{U}$, and that it has the same value as $\mathcal{U}$ at time $T^*$.*

*b) The derivatives value function $F(S, t)$ satisfies Black–Scholes partial differential equation*

$$\frac{\partial F(S,t)}{\partial t} - rF(S,t) + bS\frac{\partial F(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F(S,t)}{\partial S^2} = 0, \quad t \leq T^*. \tag{6.3}$$

**Proof:**
To simplify we proceed from the assumption that the object is a stock paying a continuous dividend yield $d$, and thus involving costs of carry $b = r - d$ with $r$ the continuous compounded risk free interest rate. Furthermore, we consider only the case where $\mathcal{U}$ is a derivative on the stock, and that $\mathcal{U}$ does not generate any payoff before time $T^*$.

We construct a portfolio consisting of $n_t = n(S_t, t)$ shares of the stock and $m_t = m(S_t, t)$ zero bonds with maturity date $T$ and a face value of $B_T = 1$. Let

$$B_t = B_T e^{-r(T-t)} = e^{-r(t-T)}$$

be the zero bond's value discounted to time $t$. We denote the time $t$ portfolio value by

$$V_t \stackrel{\text{def}}{=} V(S_t, t) = n(S_t, t) \cdot S_t + m(S_t, t) \cdot B_t.$$

It can be shown that $n_t$ and $m_t$ can be chosen so that at exercise time and respectively at maturity of $\mathcal{U}$, the portfolio value is equal to the derivative's value, i.e. $V(S_{T^*}, T^*) = F(S_{T^*}, T^*)$. Furthermore, it can be shown that the portfolio does not generate any cash flow prior to $T^*$, i.e. it is neither allowed to withdraw nor to add any money before time $T^*$. All changes in the positions must be realized by buying or selling stocks or bonds, or by means of dividend yields.

Firstly, we investigate how the portfolio value $V_t$ changes in a small period of time $dt$. By doing this, we use the notation $dV_t = V_{t+dt} - V_t$, $dn_t = n_{t+dt} - n_t$ etc.

$$\begin{aligned} dV_t &= n_{t+dt}S_{t+dt} + m_{t+dt}B_{t+dt} - n_t S_t - m_t B_t \\ &= dn_t S_{t+dt} + n_t dS_t + dm_t B_{t+dt} + m_t dB_t, \end{aligned}$$

and thus

$$dV_t = dn_t(S_t + dS_t) + n_t dS_t + dm_t(B_t + dB_t) + m_t dB_t. \tag{6.4}$$

Since the stochastic process $S_t$ is a geometric Brownian motion and therefore an Itô–process (5.10) with $\mu(x,t) = \mu x$ and $\sigma(x,t) = \sigma x$, it follows from the generalised Itô lemma (5.12) and equation (6.1)

$$dn_t = \frac{\partial n_t}{\partial t}dt + \frac{\partial n_t}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 n_t}{\partial S^2}\sigma^2 S_t^2 dt, \tag{6.5}$$

and an analogous relation for $m_t$. Using

$$(dS_t)^2 = (\mu S_t dt + \sigma S_t dW_t)^2 = \sigma^2 S_t^2 (dW_t)^2 + o(dt) = \sigma^2 S_t^2 dt + o(dt),$$

$$dB_t = rB_t dt,\ dS_t \cdot dt = o(dt) \text{ and } dt^2 = o(dt)$$

and neglecting terms of size smaller than $dt$ it follows:

$$dn_t(S_t + dS_t) = \Big(\frac{\partial n_t}{\partial t}dt + \frac{\partial n_t}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 n_t}{\partial S^2}\sigma^2 S_t^2 dt\Big)S_t + \frac{\partial n_t}{\partial S}\sigma^2 S_t^2 dt, \quad (6.6)$$

$$dm_t(B_t+dB_t) = \Big(\frac{\partial m_t}{\partial t}dt+\frac{\partial m_t}{\partial S}dS_t+\frac{1}{2}\frac{\partial^2 m_t}{\partial S^2}\sigma^2 S_t^2 dt\Big)B_t. \quad (6.7)$$

The fact that neither the derivative nor the duplicating portfolio generates any cash flow before time $T^*$ means that the terms $dn_t(S_t+dS_t)$ and $dm_t(B_t+dB_t)$ of $dV_t$ in equation (6.4) which correspond to purchases and sales of stocks and bonds have to be financed by the dividend yields. Since one share of the stock pays in a small time interval $dt$ a dividend amount of $d \cdot S_t \cdot dt$, we have

$$d \cdot n_t S_t \cdot dt = (r-b) \cdot n_t S_t \cdot dt = dn_t(S_t + dS_t) + dm_t(B_t + dB_t).$$

Substituting equations (6.6) and (6.7) in the latter equation, it holds:

$$\begin{aligned} 0 \;=\; & (b-r)n_t S_t dt + \Big(\frac{\partial m_t}{\partial t}dt + \frac{\partial m_t}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 m_t}{\partial S^2}\sigma^2 S_t^2 dt\Big)B_t \\ & + \Big(\frac{\partial n_t}{\partial t}dt + \frac{\partial n_t}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 n_t}{\partial S^2}\sigma^2 S_t^2 dt\Big)S_t + \frac{\partial n_t}{\partial S}\sigma^2 S_t^2 dt. \end{aligned}$$

Using equation (6.1) and summarising the stochastic terms with differential $dW_t$ as well as the deterministic terms with differential $dt$ containing the drift parameter $\mu$, and all other deterministic terms gives:

$$\begin{aligned} 0 \;=\; & \Big(\frac{\partial n_t}{\partial S}S_t + \frac{\partial m_t}{\partial S}B_t\Big)\mu S_t dt \\ & + \Big\{\Big(\frac{\partial n_t}{\partial t} + \frac{1}{2}\frac{\partial^2 n_t}{\partial S^2}\sigma^2 S_t^2\Big)S_t + \frac{\partial n_t}{\partial S}\sigma^2 S_t^2 \\ & + \Big(\frac{\partial m_t}{\partial t} + \frac{1}{2}\frac{\partial^2 m_t}{\partial S^2}\sigma^2 S_t^2\Big)B_t + (b-r)n_t S_t\Big\}dt \\ & + \Big(\frac{\partial n_t}{\partial S}S_t + \frac{\partial m_t}{\partial S}B_t\Big)\sigma S_t dW_t. \end{aligned} \quad (6.8)$$

This is only possible if the stochastic terms disappear, i.e.

$$\frac{\partial n_t}{\partial S}S_t + \frac{\partial m_t}{\partial S}B_t = 0. \quad (6.9)$$

Thus the first term in (6.8) is neutralized as well. Hence the middle term must also be zero:

$$\Big(\frac{\partial n_t}{\partial t} + \frac{1}{2}\frac{\partial^2 n_t}{\partial S^2}\sigma^2 S_t^2\Big)S_t + \frac{\partial n_t}{\partial S}\sigma^2 S_t^2 + \Big(\frac{\partial m_t}{\partial t} + \frac{1}{2}\frac{\partial^2 m_t}{\partial S^2}\sigma^2 S_t^2\Big)B_t + (b-r)n_t S_t = 0. \tag{6.10}$$

To further simplify we compute the partial derivative of equation (6.9) with respect to $S$ :

$$\frac{\partial^2 n_t}{\partial S^2}S_t + \frac{\partial n_t}{\partial S} + \frac{\partial^2 m_t}{\partial S^2}B_t = 0 \tag{6.11}$$

and substitute this in equation (6.10). We then obtain

$$\frac{\partial n_t}{\partial t}S_t + \frac{\partial m_t}{\partial t}B_t + \frac{1}{2}\frac{\partial n_t}{\partial S}\sigma^2 S_t^2 + (b-r)n_t S_t = 0. \tag{6.12}$$

Since the stock price $S_t$ does not depend explicitly on time, i.e. $\partial S_t/\partial t = 0$, the derivative of the portfolio value $V_t = n_t S_t + m_t B_t$ with respect to time gives:

$$\frac{\partial V_t}{\partial t} = \frac{\partial n_t}{\partial t}S_t + \frac{\partial m_t}{\partial t}B_t + m_t\frac{\partial B_t}{\partial t} = \frac{\partial n_t}{\partial t}S_t + \frac{\partial m_t}{\partial t}B_t + m_t r B_t.$$

This implies

$$\frac{\partial n_t}{\partial t}S_t + \frac{\partial m_t}{\partial t}B_t = \frac{\partial V_t}{\partial t} - r m_t B_t = \frac{\partial V_t}{\partial t} - r(V_t - n_t S_t).$$

Substituting this equation in equation (6.12) we eliminate $m_t$ and obtain

$$\frac{1}{2}\sigma^2 S_t^2\frac{\partial n}{\partial S} + \frac{\partial V_t}{\partial t} + b n_t S_t - r V_t = 0. \tag{6.13}$$

Since the zero bond value $B_t$ is independent of the stock price $S_t$, i.e. $\partial B_t/\partial S = 0$, the derivative of the portfolio value $V_t = n_t S_t + m_t B_t$ with respect to the stock price gives (using equation (6.9))

$$\frac{\partial V_t}{\partial S} = \frac{\partial n_t}{\partial S}S_t + n_t + \frac{\partial m_t}{\partial S}B_t = n_t,$$

and thus

$$n_t = \frac{\partial V_t}{\partial S}. \tag{6.14}$$

That is, $n_t$ is equal to the so–called *delta* or hedge–ratio of the portfolio (see Section 6.4.1). Since

$$m_t = \frac{V_t - n_t S_t}{B_t}$$

we can construct a duplicating portfolio if we know $V_t = V(S_t, t)$. We can obtain this function of stock price and time as a solution of the Black–Scholes differential equation

$$\frac{\partial V(S,t)}{\partial t} - rV(S,t) + bS\frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} = 0 \tag{6.15}$$

which results from substituting equation (6.14) in equation (6.13). To determine $V$ we have to take into account a boundary condition which is obtained from the fact that the cash flows at exercising time, respective maturity, i.e. at time $T^*$, of the duplicating portfolio and the derivative are equal:

$$V(S_{T^*}, T^*) = F(S_{T^*}, T^*). \tag{6.16}$$

Since the derivative has, at any time, the same cash flow as the duplicating portfolio, $F(S,t)$ also satisfies the Black–Scholes differential equation, and at any time $t \leq T^*$ it holds $F_t = F(S_t, t) = V(S_t, t) = V_t$. □

Black–Scholes' differential equation fundamentally relies on the assumption that the stock price can be modelled by a geometric Brownian motion. This assumption is only justified, if the theory building on it reproduces the arbitrage relations derived in Chapter 2. For a particular example we verify this feature. Let $V(S_t, t)$ be the value of a future contract with delivery price $K$ and maturity date $T$. The underlying object involves costs of carry at a continuous rate $b$. Since $V(S_t, t)$ depends only on the price of the underlying and time it satisfies the conditions of Theorem 6.1. From Theorem 2.1 and substituting $\tau = T - t$ for the time to maturity it follows

$$V(S,t) = Se^{(r-b)(t-T)} - Ke^{r(t-T)}.$$

Substituting the above equation into equation (6.3) it can be easily seen that it is the unique solution of Black–Scholes' differential equation with boundary condition $V(S,T) = S - K$. Hence, Black–Scholes' approach gives the same price for the future contract as the model free noarbitrage approach.

Finally, we point out that modelling stock prices by geometric Brownian motion gives reasonable solutions for short and medium time spans. Applying the model to other underlyings such as currencies or bonds is more difficult. Bond options typically have significant by longer times to maturity than stock options. Their value does not only depend on the bond price but also on interest rates which have to be considered stochastic. Modelling interest rates reasonably involves other stochastic process, which we will discuss in later chapters.

Generally exchange rates cannot be modelled by geometric Brownian motion. Empirical studies show that the performance of this model depends on the currency and on the time to maturity. Hence, applying Black–Scholes' approach to currency options has to be verified in each case. If the model is used, the foreign currency has to be understood as the option underlying with a continuous foreign interest rate $d$ corresponding to the continuous dividend yield of a stock. Thus, continuous costs of carry with rate $b = r - d$ equal the interest rate differential between the domestic and the foreign market. If the investor buys the foreign currency early, then he cannot invest his capital at home any more, and thus he loses the domestic interest rate $r$. However, he can invest his capital abroad and gain the foreign interest rate $d$. The value of the currency option results from solving Black–Scholes' differential equation (6.3) respecting the boundary condition implied by the option type.

## 6.2 Black–Scholes Formula for European Options

In this section we are going to use the Black–Scholes' equation to compute the price of European options. We keep the notation introduced in the previous chapter. That is, we denote

$$C(S,t) = C_{K,T}(S,t), \qquad P(S,t) = P_{K,T}(S,t)$$

the value of a European call respectively put option with exercise price $K$ and maturity date $T$ at time $t \leq T$, where the underlying, for example a stock, at time $t$ has a value of $S_t = S$. The value of a call option thus satisfies for all prices $S$ with $0 < S < \infty$ the differential equation

$$rC(S,t) - bS\frac{\partial C(S,t)}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} = \frac{\partial C(S,t)}{\partial t},\ 0 \leq t \leq T, \quad (6.17)$$

$$C(S,T) = \max\{0, S-K\},\ 0 < S < \infty, \quad (6.18)$$

$$C(0,t) = 0, \quad \lim_{S\to\infty} C(S,t) - S = 0,\ 0 \leq t \leq T. \quad (6.19)$$

The first boundary condition (6.18) follows directly from the definition of a call option, which will only be exercised if $S_T > K$ thereby procuring the gain $S_T - K$. The definition of geometric Brownian motion implies that the process is absorbed by zero. In other words, if $S_t = 0$ for one $t < T$ it follows $S_T = 0$. That is the call will not be exercised, which is formulated in the first part of condition (6.19). Whereas the second part of (6.19) results from the reflection that the probability of the Brownian motion falling below $K$ is fairly small once it has attained a level significantly above the exercise price.

If $S_t \gg K$ for a $t < T$ then it holds with a high probability that $S_T \gg K$. The call will be, thus, exercised and yields the cash flow $S_T - K \approx S_T$.

The differential equation (6.17) subject to boundary conditions (6.18), (6.19) can be solved analytically. To achieve this, we transform it into a differential equation known from the literature. Firstly, we substitute the time variable $t$ by the time to maturity $\tau = T - t$. By doing this, the problem with final condition (6.18) in $t = T$ changes to a problem subject to an initial condition in $\tau = 0$. Subsequently, we multiply (6.17) by $2/\sigma^2$ and replace the parameters $r, b$ by

$$\alpha = \frac{2r}{\sigma^2}, \ \beta = \frac{2b}{\sigma^2},$$

as well as the variables $\tau, S$ by

$$v = \sigma^2(\beta - 1)^2\frac{\tau}{2}, \qquad u = (\beta - 1)\log\frac{S}{K} + v.$$

While for the original parameters $0 \leq S < \infty, 0 \leq t \leq T$, for the new parameters it holds that:

$$-\infty < u < \infty, \qquad 0 \leq v \leq \frac{1}{2}\sigma^2(\beta - 1)^2 T \stackrel{\text{def}}{=} v_T.$$

Finally, we set

$$g(u, v) = e^{r\tau}C(S, T - \tau)$$

and obtain the new differential equation

$$\frac{\partial^2 g(u, v)}{\partial u^2} = \frac{\partial g(u, v)}{\partial v}. \tag{6.20}$$

with the initial condition

$$g(u, 0) = K\max\{0, e^{\frac{u}{\beta-1}} - 1\} \stackrel{\text{def}}{=} g_0(u), \quad -\infty < u < \infty. \tag{6.21}$$

Problems with initial conditions of this kind are well known from the literature on partial differential equations. They appear, for example, in modelling heat conduction and diffusion processes. The solution is given by

$$g(u, v) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi v}} g_0(\xi) e^{-\frac{(\xi-u)^2}{4v}} d\xi.$$

The option price can be obtained by undoing the above variable and parameter substitutions. In the following we denote, as in Chapter 2, by $C(S, \tau)$ the

call option price being a function of the time to maturity $\tau = T - t$ instead of time $t$. Then it holds

$$C(S,\tau) = e^{-r\tau} g(u,v) = e^{-r\tau} \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi v}} g_0(\xi) e^{-\frac{(\xi-u)^2}{4v}} d\xi.$$

Substituting $\xi = (\beta - 1)\log(x/K)$ we obtain the original terminal condition $\max\{0, x - K\}$. Furthermore, replacing $u$ and $v$ by the variables $S$ and $\tau$ we obtain

$$\begin{aligned} C(S,\tau) &= e^{-r\tau} \int_{0}^{\infty} \max(0, x-K) \frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}x} \\ &\quad \exp\left\{ -\frac{[\log x - \{\log S + (b - \frac{1}{2}\sigma^2)\tau\}]^2}{2\sigma^2\tau} \right\} dx. \end{aligned} \tag{6.22}$$

In the case of geometric Brownian motion $S_T - S_t$ is lognormally distributed, i.e. $\log(S_T - S_t)$ is normally distributed with parameters $(b - \frac{1}{2}\sigma^2)\tau$ and $\sigma^2\tau$. The conditional distribution of $S_T$ given $S_t = S$ is therefore lognormal as well but with parameters $\log S + (b - \frac{1}{2}\sigma^2)\tau$ and $\sigma^2\tau$. However, the integrant in equation (6.22) is except for the term $\max(0, x - K)$ the density of the latter distribution. Thus, we can interpret the price of a call as the discounted expected option payoff $\max(0, S_T - K)$, which is the terminal condition, given the current stock price $S$ :

$$C(S,\tau) = e^{-r\tau}\, \mathsf{E}[\max(0, S_T - K) \mid S_t = S]. \tag{6.23}$$

This property is useful when deriving numerical methods to compute option prices. But before doing that, we exploit the fact that equation (6.22) contains an integral with respect to the density of the lognormal distribution to further simplify the equation. By means of a suitable substitution we transform the term into an integral with respect to the density of the normal distribution and we obtain

$$C(S,\tau) = e^{(b-r)\tau} S\Phi(y + \sigma\sqrt{\tau}) - e^{-r\tau} K\Phi(y), \tag{6.24}$$

where we use $y$ as a abbreviation for

$$y = \frac{\log \frac{S}{K} + (b - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}. \tag{6.25}$$

$\Phi$ denotes the standard normal distribution function

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{z^2}{2}} dz.$$

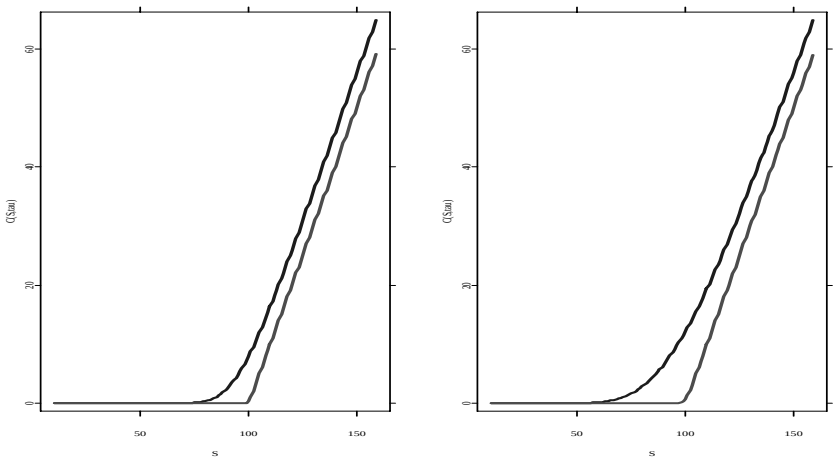



Figure 6.1: Black-Scholes prices $C(S,\tau)$ for different times to maturity $\tau = 0.6$ and $r = 0.1$ and strike price $K = 100$. Left figure $\sigma = 0.15$, right figure $\sigma = 0.3$.

SFEbsprices

Equations (6.24) and (6.25) are called the *Black–Scholes Formulae*. Figure 6.1 represents the Black-Scholes prices for the European call option $C(S,\tau)$ for different values of times to maturity $\tau$. When $\tau$ goes to 0, the price of the call is approaching to the payoff of the option. The economic reason behind this is that there is a little probability for fluctuations in the underlying price within such a small interval $\tau$.

Figure 6.2 displays the Black-Scholes price $C(S,\tau)$ as a function of $S_t$, which is modelled as a geometric Brownian motion. The figure shows that the option price moves according to the price fluctuations of the underlying.

For the limit cases $S \gg K$ and $S = 0$ it holds:

- If $S \gg K$ then $y \gg 0$ and thus $\Phi(y+\sigma\sqrt{\tau}) \approx \Phi(y) \approx 1$. It follows that the value of a call option on a non dividend paying stock, $b = r$, approaches $S - e^{-r\tau}K$. That is, it can be approximated by the current stock price minus the discounted exercise price.
- If $S = 0$ then $y = -\infty$ and therefore $\Phi(y) = 0$. Thus the option is worthless: $C(0,\tau) = 0$.

The corresponding Black–Scholes Formula for the price $P(S,\tau)$ of a Euro-

pean put option can be derived by solving Black–Scholes differential equation subject to suitable boundary conditions. However, using the put–call parity (Theorem 2.3) is more convenient:

$$P(S,\tau) = C(S,\tau) - Se^{(b-r)\tau} + Ke^{-r\tau}.$$

From this and equation (6.24) we obtain

$$P(S,\tau) = e^{-r\tau}K\Phi(-y) - e^{(b-r)\tau}S\Phi(-y-\sigma\sqrt{\tau}). \tag{6.26}$$

As we see the value of European put and call options can be computed by explicit formulae. The terms in equation (6.24) for, say the value of a call option, can be interpreted in the following way. Restricting to the case of a non dividend paying stock, $b = r$, the first term, $S\Phi(y+\sigma\sqrt{\tau})$, represents the value of the stock which the option holder obtains when he decides to exercise the option. The other term, $e^{-r\tau}K\Phi(y)$, represents the value of the exercise price. The quotient $S/K$ influences both terms via the variable $y$.

Deriving Black–Scholes' differential equation we saw, in particular, that the value of a call option had been duplicated by means of bonds and stocks. The amount of money invested in stocks was $\frac{\partial C}{\partial S}S$ with $\frac{\partial C}{\partial S}$ being the hedge ratio. This ratio, also called delta, determines the relation of bonds and stocks necessary to hedge the option position. Computing the first derivative of Black–Scholes' formula in equation (6.24) with respect to S we obtain

$$\frac{\partial C(S,t)}{\partial S} = \Phi(y+\sigma\sqrt{\tau}).$$

Thus the first term in equation (6.24) reflects the amount of money of the duplicating portfolio invested in stocks, the second term the amount invested in bonds.

### 6.2.1 Numerical Approximation

Since the standard normal distribution can be evaluated only numerically, the implementation of Black–Scholes' formula depending on the standard normal distribution requires an approximation of the latter. This approximation can have an impact on the computed option value. To illustrate, we consider several approximation formulae, see for example Hastings (1955)

**a.)** The normal distribution can be approximated in the following way:

$$\Phi(y) \approx 1 - (a_1 t + a_2 t^2 + a_3 t^3)e^{-\frac{y^2}{2}}, \text{ where } t = (1+by)^{-1} \text{ and}$$

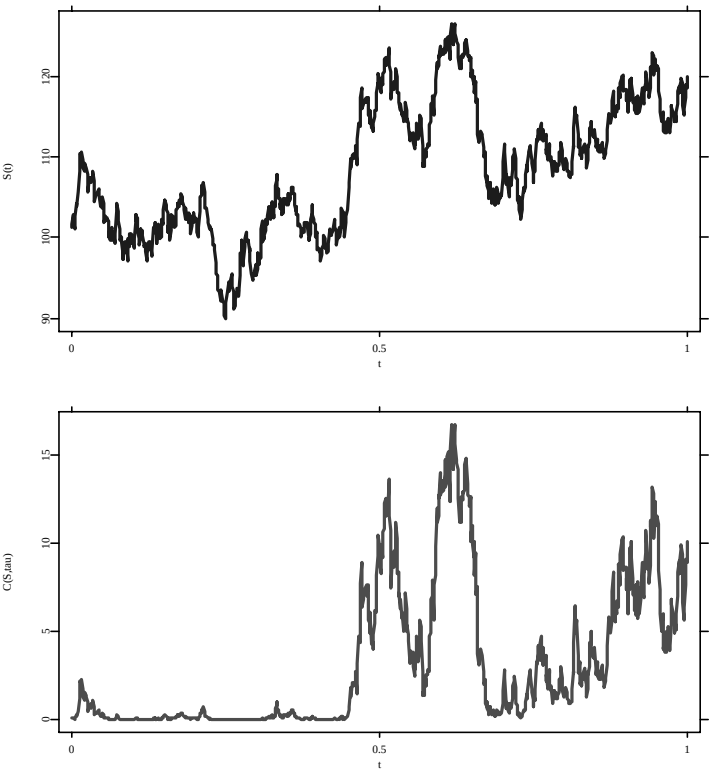



Figure 6.2: Upper panel: sample path of the price process of the underlying $S$, lower panel: Black-Scholes prices $C(S, \tau)$ for strike $K = 100$, $r = 0.05$ and expiry at $T = 1$ where the initial value of the underlying is taken from the above sample path.

SFEbsbm

$$\begin{aligned} b &= 0.332672527, \\ a_1 &= 0.17401209, \\ a_2 &= -0.04793922, \\ a_3 &= 0.373927817. \end{aligned}$$

The approximating error is independent of $y$ of size $10^{-5}$.

SFENormalApprox1

**b.)**

$\Phi(y) \approx 1 - (a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5) e^{-\frac{y^2}{2}}$, where $t = (1 + by)^{-1}$ and

$$\begin{aligned} b &= 0.231641888, \\ a_1 &= 0.127414796, \\ a_2 &= -0.142248368, \\ a_3 &= 0.71070687, \\ a_4 &= -0.726576013, \\ a_5 &= 0.530702714. \end{aligned}$$

The error of this approximation is of size $10^{-7}$. SFENormalApprox2

**c.)** An approximation of the normal distribution, with error size $10^{-5}$ is given by:

$$\Phi(y) \approx \begin{cases} 0.5 - s, & \text{if } y < 0 \\ 0.5 + s, & \text{else} \end{cases}$$

where $t = |y|$,

$$s = \frac{1}{2} - \frac{1}{2(1 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5)^8}$$

and

$$\begin{aligned} a_1 &= 0.099792714, \\ a_2 &= 0.044320135, \\ a_3 &= 0.009699203, \\ a_4 &= -0.000098615, \\ a_5 &= 0.00581551. \end{aligned}$$

SFENormalApprox3

| $x$ | norm-a | norm-b | norm-c | norm-d | iter |
|---|---|---|---|---|---|
| 1.0000 | 0.8413517179 | 0.8413447362 | 0.8413516627 | 0.8413441191 | 6 |
| 1.1000 | 0.8643435425 | 0.8643338948 | 0.8643375717 | 0.8643341004 | 7 |
| 1.2000 | 0.8849409364 | 0.8849302650 | 0.8849298369 | 0.8849309179 | 7 |
| 1.3000 | 0.9032095757 | 0.9031994476 | 0.9031951398 | 0.9031993341 | 8 |
| 1.4000 | 0.9192515822 | 0.9192432862 | 0.9192361959 | 0.9192427095 | 8 |
| 1.5000 | 0.9331983332 | 0.9331927690 | 0.9331845052 | 0.9331930259 | 9 |
| 1.6000 | 0.9452030611 | 0.9452007087 | 0.9451929907 | 0.9452014728 | 9 |
| 1.7000 | 0.9554336171 | 0.9554345667 | 0.9554288709 | 0.9554342221 | 10 |
| 1.8000 | 0.9640657107 | 0.9640697332 | 0.9640670474 | 0.9640686479 | 10 |
| 1.9000 | 0.9712768696 | 0.9712835061 | 0.9712842148 | 0.9712839202 | 11 |
| 2.0000 | 0.9772412821 | 0.9772499371 | 0.9772538334 | 0.9772496294 | 12 |

Table 6.1: Several approximations to the normal distribution

**d.)** Finally we present the Taylor expansion:

$$\begin{aligned}\Phi(y) &\approx \frac{1}{2}+\frac{1}{\sqrt{2\pi}}\left(y-\frac{y^3}{1!2^1 3}+\frac{y^5}{2!2^2 5}-\frac{y^7}{3!2^3 7}+\cdots\right)\\ &= \frac{1}{2}+\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{y^{2n+1}}{n!2^n(2n+1)}.\end{aligned}$$

By means of this series the normal distribution can be approximated arbitrarily close, depending on the number of terms used in the summation. Increasing the number of terms increases the number of arithmetic operations as well.

SFENormalApprox4

Table 6.1 compares all four approximation formulae. The Taylor series was truncated after the first term where the absolute value was smaller than $10^{-5}$. The last column shows the number of terms used.

Table 6.2 shows the price of a particular European call option computed by means of the four approximations presented above.

## 6.3 Simulation

Simulation techniques are essential tools of financial engineering. Their performance depends decisively on the quality of random numbers used, and today one cannot reply on random numbers generators in software package to be satisfactory in every respect. For applications, it should be checked that

| | | | | |
|---|---|---|---|---|
| Stock price $S_t$ | | 230.00 | EUR | |
| Exercise price $K$ | | 210.00 | EUR | |
| Time to maturity $\tau = T - t$ | | 0.50000 | | |
| Continuous interest rate $r$ | | 0.04545 | | |
| Volatility $\sigma$ | | 0.25000 | | |
| No dividends | | | | |
| | norm-a | norm-b | norm-c | norm-d |
| Option prices | 30.74262 | 30.74158 | 30.74352 | 30.74157 |

Table 6.2: Prices of a European call option for different approximations of the normal distribution SFEBSCopt1

the generated random numbers reflect all the important features, depending on the particular application. Below, we give a short discussion of simulating random variables.

### 6.3.1 Linear Congruential Generator

One of the most common pseudo random number generators is the linear congruential generator which uses a recurrence scheme to generate numbers:

$$N_i = (aN_{i-1} + b) \mod M \tag{6.27}$$
$$U_i = N_i/M \tag{6.28}$$

where $N_i$ is the sequence of pseudo random numbers and $(a, b, M)$ are generator-specific integer constants. mod is the modulo operation, $a$ the multiplier and $b$ the increment, $a, b, N_0 \in 0, 1, \ldots, M-1$ with $a \neq 0$.

The linear congruential generator starts choosing an arbitrary seed $N_0$ and will always produce an identical sequence from that point on. The maximum amount of different numbers the formula can produce is the modulus $M$. The pseudo random variables $N_i/M$ are uniformly distributed.

The period of a general linear congruential generator $N_i$ is at most $M$, but in most cases it is less than that. The period should be large in order to ensure randomness, otherwise a small set of numbers can make the outcome easy to forecast. It may be convenient to set $M = 2^{32}$, since this makes the computation of $aN_{i-1} + b \mod M$ quite efficient.

In particular, $N_0 = 0$ must be ruled out in case $b = 0$, otherwise $N_i = 0$ would repeat. If $a = 1$, the sequence is easy to forecast and the generated sets are:

$$N_n = (N_0 + nb) \mod M$$

The linear congruential generator will have a full period if, Knuth (1997):

1. $b$ and $M$ are prime.
2. $a - 1$ is divisible by all prime factors of $M$.
3. $a - 1$ is a multiple of 4 if $M$ is a multiple of 4.
4. $M >\max(a, b, N_0)$.
5. $a > 0, b > 0$.

Exactly, when the period is $M$, a grid point on a lattice over the interval [0,1] with size $\frac{1}{M}$ is occupied once.

Once the $N_i$ numbers are generated, they can be arranged in $m$-tupels $(N_i, N_{i+1}, ..., N_{i+m-1})$ for $i \geq 1$. The corresponding points $(U_i, ..., U_{i+m-1}) \in [0, 1]^m$ will lie on $(m - 1)$ dimensional hyperplanes (parallel straight lines for $m = 2$). Analysing the case for two planes or $m = 2$, where the distance between planes is large, the sequence is:

$$N_i = (aN_i + b) \mod M = aN_{i+1} + b - kM$$

for $kM \leq aN_{i-1} + b < (k + 1)M$.

For all integers $Z_0, Z_1$:

$$\begin{aligned} Z_0N_{i-1} + Z_1N_i &= Z_0N_{i-1} + Z_1(aN_{i-1} + b - kM) \\ &= N_{i-1}(Z_0 + aZ_1) + Z_1b - Z_1kM \\ &= M(N_{i-1}\frac{Z_0 + aZ_1}{M} - Z_1k) + Z_1b \end{aligned}$$

Hence

$$Z_0U_{i-1} + Z_1U_i = c + Z_1bM^{-1} \tag{6.29}$$

Therefore, for a given tupel $(Z_0, Z_1)$ several parallel straight lines in the $(U_{i-1}, U_i)$ plane are defined, one for each $c = c(i)$. Moreover, the distribution of the random numbers will be subject to the minimum number of hyperplanes or the maximum distance between them. That means, if a tupel exists $(Z_0, Z_1)$ with only few of its straight lines cutting the square $[0, 1]^2$, then there will be areas of the square without random points, which will not satisfy the condition of a uniform distribution of the points in the unit square.

Notice that the variable $c$ is an integer since only integer tupels are admitted and they require

$$Z_0 + aZ_1 = 0 \mod M$$

The number of straight lines will be defined by solving $c$ in equation (6.29). Applying $U_i \sim U[0,1)$ an interval $c_{min} \leq c \leq c_{max}$ is obtained. For each $c$ in that interval, its straight line cuts the square $[0,1]^2$.

**Example 6.1** *Consider the sequence* $N_i = 2N_{i-1} \mod 11$*, with values* $a = 2, b = 0, M = 11$*. The solutions to equation (6.29) are* $Z_0 = -2$ *and* $Z_1 = 1$*. The family of straight lines in the* $(U_{i-1}, U_i)$ *plane looks like:*

$$-2U_{i-1} + U_i = c$$

*For* $U_i \sim U[0,1]$*, the variable* $c$ *is defined over the interval* $(-2,1)$*. Since* $c$ *is an integer, only* $c = -1$ *and* $c = 0$ *hold and cut the interior of the square* $[0,1)^2$*. Figure 6.3 shows that the points generated by the algorithm are lying on the straight lines with* $c = -1$ *and* $c = 0$*, whose points form a lattice.*

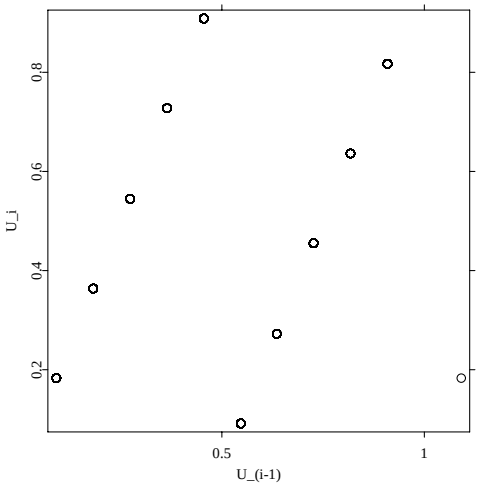



Figure 6.3: The scatterplot of $(U_{i-1}, U_i)$

SFErangen1

**Example 6.2** *Suppose* $N_i = 1229N_{i-1} \mod 2048$*. The values of the tupel* $(Z_0, Z_1)$ *are* $Z_0 = -1$ *and* $Z_1 = 5$*, because*

$$-1 + 1229(5) = 6144 = 3(2048)$$

*The family of straight lines in the* $(U_{i-1}, U_i)$ *plane is*

$$-1U_{i-1} + 5U_i = c$$

*and the distance between the straight lines over the vertical* $U_i$ *axis is* $\frac{1}{Z_1} = \frac{1}{5}$. *Figure 6.4 shows that all the points* $(U_{i-1}, U_i)$ *generated are lying on six straight lines, with* $c \in -1, 0, 1, 2, 3, 4$.

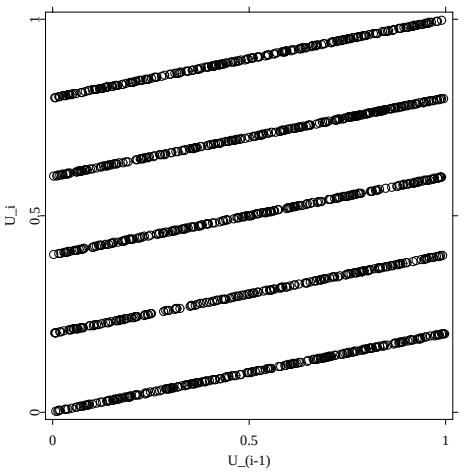



Figure 6.4: The scatterplot of $(U_{i-1}, U_i)$

SFErangen2

Pseudo random numbers produced by linear congruential generators are extremely sensitive to the choice of $a$, $b$ and $M$. Historically, poor choices had ruined implementations of linear congruential generators. A famous example of this is RANDU, the official IBM $U[0, 1]$ generator for years. It is considered to be one of the most ill-conceived random number generators. The RANDU generator is defined as

$$N_i = aN_{i-1} \mod M$$

with $a = 2^{16} + 3$, $M = 2^{31}$ and $N_0$ odd. The defined values were chosen, because with a 32 bit integer word size the arithmetic of mod $2^{31}$ and $2^{16} + 3 = 65539$ calculations could be done quickly, see Knuth (1997).

This generator fails the spectral test for dimensions greater than 2, which is a quality test of a linear congruential generators. It is based on the fact that

if a linear congruential generator is used to choose points in an $n$-dimensional space, triples of points will lie on, at most, $M^{\frac{1}{n}}$ hyperplanes. This is due to serial correlation between successive values of the sequence $N_i$. As a result of the wide use of RANDU in the early 70's many simulation studies from that time are seen as suspicious, Press, Teukolsky and Vetterling (1992).

To show the problem with the defined values consider the previous example, but this time every term should be taken with $\mod 2^{31}$. After recursion, the random points in the cube $[0,1]^3$ lie on only 15 planes. Figure 6.5 shows the scatterplot $(U_{i-2}, U_{i-1}, U_i)$ for the previous example using RANDU generator.

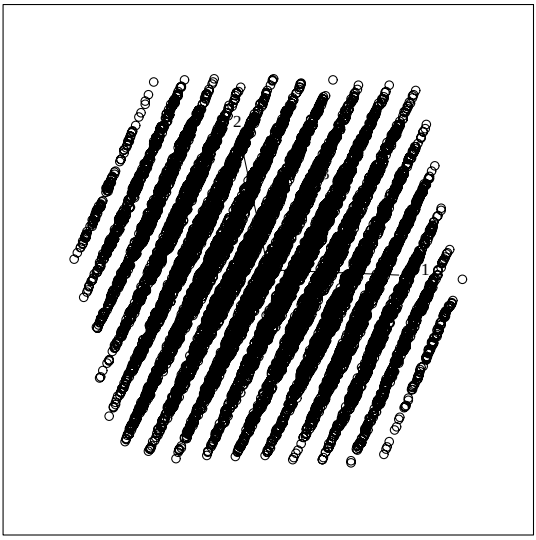


Figure 6.5: The scatterplot of $(U_{i-2}, U_{i-1}, U_i)$

SFErandu

In practice, many common linear congruential generators fail statistical significance tests when exhibiting shorter than expected periods for some seed states, poor dimensional distribution, dependence between successive values, some bits being more random than others and lack of uniformity. A disadvantage of the linear congruential generators is the restrictedness of the period $M$, which is limited to 32 bit size. However, this can be corrected by *shuffling* the random numbers in a random way and the period will get close to infinity.

### 6.3.2 Fibonacci Generators

Another example of pseudo random number generators are the Fibonacci generators, whose aim is to improve the standard linear congruential generator. These are based on a Fibonacci sequence:

$$N_{i+1} = N_i + N_{i-1} \mod M$$

This recursion formula is related to the Golden ratio. The ratio of consecutive Fibonacci numbers $\frac{F(n+1)}{F(n)}$ converges to the golden ratio $\gamma$ as the limit, defined as one solution equal to $\frac{1+\sqrt{5}}{2} = 1.6180$ of the equation $x = 1 + \frac{1}{x}$.

The original formula is a three term recursion, which is not appropriate for generating random numbers. The modified approach, the lagged Fibonacci generator is defined as

$$N_{i+1} = N_{i-\nu} + N_{i-\mu} \mod M$$

for any $\nu, \mu \in \mathbb{N}$.

The quality of the outcome for this algorithm is sensitive to the choice of the initial values, $\nu$ and $\mu$. Any maximum period of the lagged Fibonacci Generator has a large number of different possible cycles. There are methods where a cycle can be chosen, but this might endanger the randomness of future outputs and statistical defects may appear.

**Example 6.3** *Let $\nu = 17$ and $\mu = 5$ be values in $\mathbb{N}$, the lagged Fibonnaci generator is:*

$$U_i = (U_{i-17} - U_{i-5})$$

*if $U_i < 0$ then set $U_i = U_i + 1.0$.*

*This recursion produces random numbers $U_i \in [0, 1)$ and requires the generation of the initial seventeen uniforms by a congruential generator. The algorithm for Fibonacci generators with seed $N_0$, equal to any number, is simplified as follows:*

*Repeat:*

$$\zeta = U_i - U_j$$
$$\text{if } \zeta < 0\text{: } \zeta = \zeta + 1$$
$$U_i = \zeta$$
$$i = i - 1$$

$j = j - 1$

if $i = 0$: $i = 17$

if $j = 0$: $j = 17$

*Figure 6.6 shows 10000 random points generated with the Fibonacci Algorithm, where the initial uniforms random variables $U_1 \dots U_{17}$ were calculated with a linear congruential generator with values M=714025, a=1366, b=150889. Notice that the points show a random structure.*

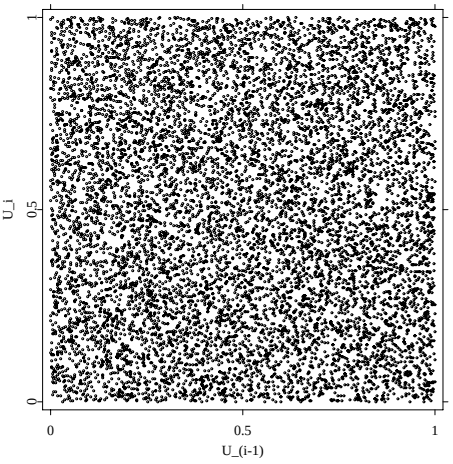



Figure 6.6: The scatterplot of $(U_{i-1}, U_i)$

SFEfibonacci

### 6.3.3 Inversion Method

Many programming languages can generate pseudo-random numbers which are distributed according to the standard uniform distribution and whose probability is the length $b - a$ of the interval $(a, b) \in (0, 1)$. *The inverse method* is a method of sampling a random number from any probability distribution, given its cumulative distribution function (cdf).

Suppose $U_i \sim U[0, 1]$ and $F(x)$ a strictly increasing continuous distribution then $X_i \sim F$, if $X_i = F^{-1}(U_i)$.

Proof:

$$P(X_i \leq x) = P\left\{F^{-1}(U_i) \leq x\right\} = P\left\{U_i \leq F(x)\right\} = F(x)$$

Usually $F^{-1}$ is often hard to calculate, but the problem can be solved using transformation methods. Suppose that $X$ is a random variable with the density function $f(x)$ and the distribution function $F(x)$. Further assume $h$ be strictly monotonous, then $Y = h(X)$ has the distribution function $F\left\{h^{-1}(y)\right\}$. If $h^{-1}$ is continuous, then for all $y$ the density of *h(X)* is, Härdle and Simar (2007):

$$f_Y(y) = f_X\left\{h^{-1}(y)\right\}\left|\frac{dh^{-1}(y)}{dy}\right|$$

**Example 6.4** *Apply the transformation method in the exponential case. The density of an exponential function is* $f_Y(y) = \lambda \exp\{-\lambda y\} I(y \geq 0)$ *with* $\lambda \geq 0$*, and its inverse is equal to* $h^{-1}(y) = \exp\{-\lambda y\}$ *for* $y \geq 0$*. Define* $y = h(x) = -\lambda^{-1} \log x$ *with* $x > 0$*. We would like to know whether* $X \sim U[0,1]$ *leads to an exponentially distributed random variable* $Y \sim \exp(\lambda)$*.*

*Using the definition of the transformation method, we have*

$$f_Y(y) = f_X\left\{h^{-1}(y)\right\}\left|\frac{dh^{-1}(y)}{dy}\right| = |(-\lambda)\exp\{-\lambda y\}| = \lambda \exp\{-\lambda y\}$$

*Hence* $f_Y(y)$ *is exponentially distributed.*

### 6.3.4 Box-Muller Method

The Box-Muller method allows us to transform uniformly distributed random variables to a set of independent standard normally distributed random numbers. Let the unit square be $S = [0,1]^2$ and $f_X(x) = 1$ the density of the uniform distribution. The most basic form of the transformation looks like:

$$y_1 = \sqrt{-2\log x_1}\cos 2\pi x_2 = h_1(x_1, x_2)$$

$$y_2 = \sqrt{-2\log x_2}\sin 2\pi x_2 = h_2(x_1, x_2)$$

where $h(x)$ is defined on the square $[0,1]^2$. Notice that when $x_1$ is very close to zero, the transformation can have numerical stability problems, especially during the generation of numbers in stochastic modelling.

In this method we start with two independent random numbers, $x_1$ and $x_2$, which are uniformly distributed over (0,1). Then, we apply the above

transformations to get two new independent random numbers, which have a normal distribution with a zero mean and a standard deviation of one.

Following the transformation method approach, the inverse function $h(x)^{-1}$ is given by

$$h^{-1}(x) = \begin{cases} x_1 = \exp\left\{-\frac{1}{2}(y_1^2 + y_2^2)\right\} \\ x_2 = (2\pi)^{-1} \arctan \frac{y_2}{y_1} \end{cases}$$

The determinant of the Jacobian matrix is

$$|Jacobian| = \det \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{pmatrix}$$

with

$$\frac{\partial x_1}{\partial y_1} = \exp\left\{-\frac{1}{2}(y_1^2 + y_2^2)\right\}(-y_1)$$

$$\frac{\partial x_1}{\partial y_2} = \exp\left\{-\frac{1}{2}(y_1^2 + y_2^2)\right\}(-y_2)$$

$$\frac{\partial x_2}{\partial y_1} = \frac{1}{2\pi}\left(\frac{1}{1 + \frac{y_2^2}{y_1^2}}\right)\frac{y_2}{y_1^2}$$

$$\frac{\partial x_2}{\partial y_2} = \frac{1}{2\pi}\left(\frac{1}{1 + \frac{y_2^2}{y_1^2}}\right)\frac{1}{y_1}$$

Then,

$$|Jacobian| = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}(y_1^2 + y_2^2)\right\}\left(-y_1\frac{1}{1 + \frac{y_2^2}{y_1^2}}\frac{1}{y_1} - y_2\frac{1}{1 + \frac{y_2^2}{y_1^2}}\frac{y_2}{y_1^2}\right)$$

$$= -\frac{1}{2\pi}\exp\left\{-\frac{1}{2}(y_1^2 + y_2^2)\right\}$$

The last equation states that the determinant of the Jacobian is the density of the standard normal distribution in $\mathbb{R}^2$. The components of the vector $Y$ are independent since this density is the product of two univariate densities. Hence $(Y_1, Y_2) \sim N_2(0, I_2)$.

The next Box-Muller Algorithm simplifies the previous results.

1. $U_1 \sim U[0,1], U_2 \sim U[0,1]$
2. $\Theta = 2\pi U_2, \rho = \sqrt{-2\log U_1}$
3. $z_1 = \rho\cos\theta$ is $N(0,1)$
   $z_2 = \rho\sin\theta$ is $N(0,1)$

Summarizing, the Box-Muller algorithm provides two standard normal distributions, when the components of the vector $X$ are $\sim U[0,1]$. Figure 6.7 shows the results.

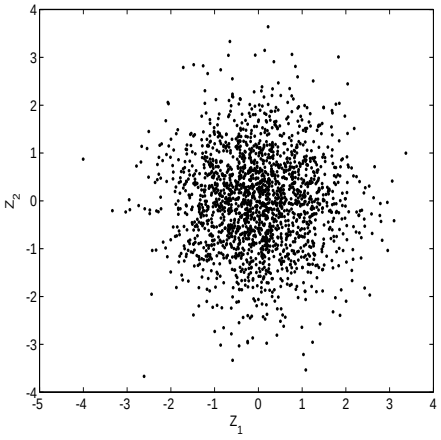



Figure 6.7: The scatterplot of $(z_1, z_2)$

SFEbmuller

## 6.3.5 Variant of Marsaglia Method

The variant of Marsaglia method is the polar form of the Box-Muller transformation. It is faster and more robust since it avoids the calculation of the trigonometric functions in the Box-Muller method.

This method generates uniformly distributed random values $V_1, V_2$ over the interval [-1,1] by transforming $U[0,1]$ variables. Two values $(V_1, V_2)$ define a point in the $(V_1, V_2)$ plane. The points will be accepted if $V_1^2 + V_2^2 < 1$ i.e. $(V_1, V_2)$ are uniformly distributed on the unit circle with the density $f(V_1, V_2) = \pi^{-1}$.

The transformation that maps the coordinates of the unit circle into the unit

square $\mathrm{S} = [0,1]^2$ is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} V_1^2 + V_2^2 \\ 2\pi^{-1} \arctan \frac{V_2}{V_1} \end{pmatrix}$$

Hence $(x_1, x_2)$ is uniformly distributed on S.

Using the previous results as inputs in the Box-Muller method, the following relationships hold

$$\cos 2\pi x_2 = \frac{V_1}{\sqrt{V_1^2 + V_2^2}}$$
$$\sin 2\pi x_2 = \frac{V_2}{\sqrt{V_1^2 + V_2^2}}$$

and it is no longer necessary to evaluate trigonometric functions.

The Marsaglia (Polar method) Algorithm is described by

1. $U_1, U_2 \sim U[0,1]$; $V_i = 2U_i - 1$ with $W = V_1^2 + V_2^2 < 1$
2. $Z_1 = V_1 \sqrt{-2\log(W)/W} \sim N(0,1)$
   $Z_2 = V_2 \sqrt{-2\log(W)/W} \sim N(0,1)$

The Marsaglia Polar method, which is a type of rejection sampling, is the polar equivalent of the Box-Muller method. Typically it is faster, because it uses only one transcendental function instead of at least two, even though it throws away $1 - \frac{\pi}{4} \approx 21.46\%$ of the total input uniformly distributed random number pairs per normal pair variable generated, requiring $\frac{4}{\pi} \approx 1.2732$ input random numbers per output random number.

## 6.4 Risk Management and Hedging

Trading options is particularly risky due to the possibly high random component. Advanced strategies to reduce and manage this risk can be derived from the Black–Scholes formula (6.24). To illustrate this issue we consider an example and some traditional strategies.

**Example 6.5**
*A bank sells a European call option to buy 100 000 shares of a non dividend paying stock for 600 000* EUR. *The details of this option are given in Table 6.3.*

*Applying Black–Scholes' formula (6.24) for a non dividend paying stock,* $b = r$, *gives a theoretical value of 480 119* EUR, *approximately 480 000* EUR,

| | |
|---|---|
| Current time $t$ | 6 weeks |
| Maturity $T$ | 26 weeks |
| Time to maturity $\tau = T - t$ | 20 weeks = 0.3846 |
| Continuous annual interest rate $r$ | 0.05 |
| Annualized stock volatility $\sigma$ | 0.20 |
| Current stock price $S_t$ | 98 EUR |
| Exercise price $K$ | 100 EUR |

Table 6.3: Data of the example

*of the above option. That is, the bank sold the option about 120 000* EUR *above its theoretical value. But it takes the risk to incur substantial losses.*

*A strategy to manage the risk due to the option would be to do nothing, i.e. to take a* naked position*. Should the option be exercised at maturity the bank has to buy the shares for the stock price prevailing at maturity. Assume the stock trades at* $S_T = 120$ EUR*. Then an options' exercise costs the bank* $100\,000 \cdot (S_T - K) = 2\,000\,000$ EUR*, which is a multiple of what the bank received for selling the derivative. However, if the stock trades below* $K = 100$ EUR *the option will not be exercised and the bank books a net gain of 600 000* EUR*.* *SFEBSCopt2*

*In contrast to the naked position, it is possible to set up a* covered position *by buying 100 000 shares at* $100\,000 \cdot S_t = 9\,800\,000$ EUR *at the same time the option is sold. In case* $S_T > K$ *the option will be exercised and the stocks will be delivered at a price of* $100\,000 \cdot K = 10\,000\,000$ EUR*, which discounted to time* $t$ *is about 9 800 000* EUR*. Thus the bank's net gain is equal to 600 000* EUR*, the price at which the option is sold. If the stock price decreases to* $S_T = 80$ EUR *then the option will not be exercised. However, the bank incurs a loss of 2 000 000* EUR *due to the lower stock price, which is as above a multiple of the option price. Note that from put–call parity for European options (Theorem 2.3) it follows that the risk due to a covered short call option position is identical to the risk due to naked long put option position.*

*Both risk management strategies are unsatisfactory because the cost varies significantly between 0 and large values. According to Black–Scholes the option costs on average around 480 000* EUR*, and a perfect hedge eliminates the impact of random events such that the option costs exactly this amount.*

An expensive hedging strategy, i.e. a strategy to decrease the risk associated with the sale of a call, is the so–called *stop–loss strategy*: The bank selling

the option takes an uncovered position as long as the stock price is below the exercise price, $S_t < K$, and sets up a covered position as soon as the call is in–the–money, $S_t > K$.

The shares to be delivered in case of options exercise are bought as soon as the stock $S_t$ trades above the exercise price $K$, and are sold as soon as $S_t$ falls below the exercise price $K$.

Since all stocks are sold and bought at $K$ after time 0 and at maturity $T$ either the stock position is zero, $(S_t < K)$, or the stocks are sold at $K$ to the option holder, $(S_t > K)$, this strategy bears no costs.

Note that playing a stop–loss strategy bears a cost if $S_0 > K$, i.e. stocks are bought at $S_0$ and sold at $K$ :

$$\text{costs of a stop–loss hedging strategy: } \max(S_0 - K,\ 0).$$

Because these costs are smaller than the Black–Scholes price $C(S_0, T)$ arbitrage would be possible by running a stop–loss strategy. However, this reasoning ignores some aspects:

- Buying and selling stocks bear transaction costs,
- Buying stocks before time $T$ involves binding capital leading to renounce of interest rate revenue,
- practically it is not possible to buy or sell stocks exactly at $K$ rather stocks are bought at $K + \delta$ if stocks are increasing and stocks are sold at $K - \delta$ if stocks are decreasing, for a $\delta > 0$.

In practice, purchases and sales take place only after $\Delta t$ time units. The larger $\Delta t$, the greater $\delta$ in general, and the less transaction costs have to be paid. Hull (2000) investigated in a Monte Carlo study with $M = 1000$ simulated stock price paths the stop–loss strategy's ability to reduce the risk associated with the sale of a call option. For each simulated path the costs $\Lambda_m, m = 1, ..., M$, caused by applying the stop–loss strategy are registered and their sample variance

$$\hat{v}_\Lambda^2 = \frac{1}{M} \sum_{m=1}^{M} (\Lambda_m - \frac{1}{M} \sum_{j=1}^{M} \Lambda_j)^2$$

is computed. Dividing the sample standard deviation by the call price measures the remaining risk of the stop–loss hedged short call position

$$L = \frac{\sqrt{\hat{v}_\Lambda^2}}{C(S_0, T)}.$$

| $\Delta t$ (weeks) | 5 | 4 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ |
|---|---|---|---|---|---|---|
| $L$ | 1.02 | 0.93 | 0.82 | 0.77 | 0.76 | 0.76 |

Table 6.4: Performance of the stop–loss strategy

Table 6.4 shows the results. A perfect hedge would reduce the risk to zero, i.e. $L = 0$.

In order to apply the concept of risk neutral valuation (see Cox and Ross (1976)) the probability measure has to be transformed such that the price process under this new measure is a martingale. By doing this the absence of arbitrage opportunities is guaranteed. In incomplete markets, however, a multitude of such transformations exist, see Harrison and Kreps (1979). In contrast to complete markets the trader cannot build up a self–financing portfolio reproducing the options payoff at maturity when the market is incomplete. Therefore hedging is no longer risk free, and option prices depend on risk preferences. In this context we want to point out that the lack of a perfect hedge is of great importance in practice.

### 6.4.1 Delta Hedging

In order to reduce the risk associated with option trading, more complex hedging strategies than those considered so far are applied. Let us look at the following example. Sell a call option on a stock, and try to make the value of this portfolio for small time intervals as insensitive as possible to small changes in the price of the underlying stock. This is what is called delta hedging. Later on, we consider further *Greeks* (gamma, theta, vega, etc.) to fine tune the hedged portfolio.

By the *delta* or the *hedge ratio* we understand the derivative of the option price with respect to the stock price. In a discrete time model we use the differential quotient of the change in the option price $\Delta C$ with respect to a change in the stock price $\Delta S$ :

$$\Delta = \frac{\partial C}{\partial S} \quad \text{oder} \quad \Delta = \frac{\Delta C}{\Delta S}.$$

The delta of other financial instruments is defined accordingly. The stock itself has the value $S$. Consequently it holds that $\Delta = \partial S/\partial S = 1$. A futures contract on a non dividend paying stock has a value of $V = S - K \cdot e^{-r\tau}$ (see Theorem 2.1) and thus its delta is $\Delta = \partial V/\partial S = 1$ as well. Stocks and future

contracts can therefore be used equivalently in delta hedging strategies. If the latter are available they are preferable due to lower transaction costs.

**Example 6.6**
*A bank sells calls on 2000 shares of a stock for a price of* $C = 10$ EUR*/share at a stock price of* $S_0 = 100$ EUR*/share. Let the call's delta be* $\Delta = 0.4$*. To hedge the sold call options* $\Delta \cdot 2000 = 800$ *shares of the stock are added to the portfolio. Small changes in the option value will be offset by corresponding changes in the value of the portfolio's stock shares. Should the stock price increase by 1* EUR*, i.e. the value of the stock position in the portfolio increases by 800* EUR*, the value of one call on 1 share increases by* $\Delta C = \Delta \cdot \Delta S = 0.4$ EUR *and following the value of the portfolio's short call position decreases by 800* EUR*. That is, gains and losses offset because the delta of the option position is neutralised by the delta of the stock position. The portfolio has a* $\Delta = 0$*, and the bank takes a* delta neutral position*.*

Since the delta of an option depends on the stock price and time, among others, the position is only delta neutral for a short period of time. In practice, the portfolio has to be *re–balanced* frequently in order to adapt to the changing environment. Strategies to manage portfolio risk which involve frequent re–balancing are known as *dynamic hedging*. We point out that the Black–Scholes differential equation (6.3) can be derived by means of a dynamic hedge portfolio whose position is kept continuously delta neutral. This approach is analogous to reproducing the option by a duplicating portfolio.

**Example 6.7**
*In continuation of Example 6.6, suppose that the underlying stock rises within a week to 110* EUR*. Due to the time delay and the increased stock price, the option delta increases to* $\Delta = 0.5$*. In order to re-obtain a delta neutral position* $(0.5 - 0.4) \cdot 2000 = 200$ *shares of the stock have to be bought.*

From the Black–Scholes formulae for the value of European call and put options on non-dividend paying stocks it follows for the delta that:

$$\begin{aligned} \Delta = \frac{\partial C}{\partial S} &= \Phi(y + \sigma\sqrt{\tau}) \qquad (6.30) \\ \text{bzw. } \Delta = \frac{\partial P}{\partial S} &= \Phi(y + \sigma\sqrt{\tau}) - 1, \end{aligned}$$

with $y$ being defined in equation (6.25).

Figure 6.8 displays the delta (6.30) as a function of time and stock price. For an increasing stock price delta converges to 1, for decreasing stock prices it

converges to 0. Put differently, if the option is deep in–the–money (ITM) it will be exercised at maturity with a high probability. That is the reason why the seller of such an option should be long in the underlying to cover the exercise risk. On the other hand, if the option is far out–of–the–money it will probably not be exercised, and the seller can restrict themself to holding a smaller part of the underlying.

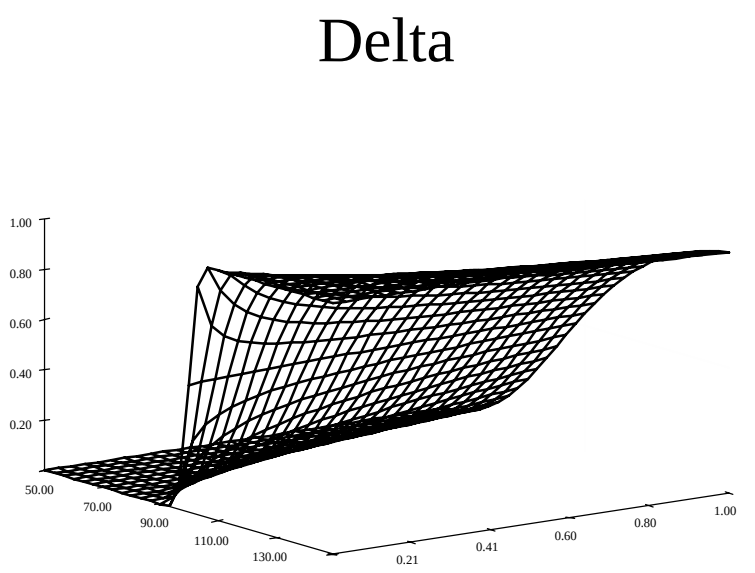



Figure 6.8: Delta as a function of the stock price (right axis) and time to maturity (left axis). SFEdelta

Furthermore, the probability $p$ that an out–of–the–money (OTM) option will be exercised and an ITM option will not be exercised at maturity is higher the longer the time to maturity. This explains why the delta for longer times to maturity becomes more flat (linear).

Table 6.5 according to Hull (2000) shows (in the same spirit as Table 6.4) the performance of the delta hedging strategy contingent on the time increments $\Delta t$ between re–balancing trades. If $\Delta t$ is small enough the risk associated with a sold call option can be managed quite well. In the limit $\Delta t \to 0$ continuous re–balancing underlying the derivation of the Black–Scholes formula follows, and the risk is perfectly eliminated ($L = 0$). The linearity of the mathematical derivative implies for the delta $\Delta_p$ of a portfolio consisting of $w_1, \ldots, w_m$ contracts of $m$ financial derivatives $1, \ldots, m$ with deltas

| $\Delta t$ (weeks) | 5 | 4 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ |
|---|---|---|---|---|---|---|
| $L$ | 0.43 | 0.39 | 0.26 | 0.19 | 0.14 | 0.09 |

Table 6.5: Performance of the delta–hedging strategy

$\Delta_1, \ldots, \Delta_m$ :

$$\Delta_p = \sum_{j=1}^{m} w_j \Delta_j.$$

**Example 6.8**
*Consider a portfolio consisting of the following* USD *options*

1. *200 000 bought calls (long position) with exercise price 1.70* EUR *maturing in 4 months. The delta of an option on 1* USD *is* $\Delta_1 = 0.54$.
2. *100 000 written calls (short position) with exercise price 1.75* EUR *maturing in 6 months and a delta of* $\Delta_2 = 0.48$.
3. *100 000 written puts (short position) with exercise price 1.75* EUR *maturing in 3 months with* $\Delta_3 = -0.51$.

*The portfolio's delta is (increases in values of written options have a negative impact on the portfolio value):*

$$\begin{aligned} \Delta_p &= 200\,000 \cdot \Delta_1 - 100\,000 \cdot \Delta_2 - 100\,000 \cdot \Delta_3 \\ &= 111\,000 \end{aligned}$$

*The portfolio can be made delta neutral by selling 111 000* USD *or by selling a corresponding future contract on* USD *(both have a delta of* $\Delta = 1$*).*

## 6.4.2 Gamma and Theta

Using the delta to hedge an option position, the option price is locally approximated by a function which is linear in the stock price $S$. Should the time $\Delta t$ elapsing until the next portfolio re–balancing not be very short, then this approximation would no longer be adequate (see Table 6.5). That is why a more accurate approximation, the Taylor expansion of $C$ as a function of $S$ and $t$, is considered:

$$\Delta C = C(S+\Delta S,\, t+\Delta t) - C(S,t) = \frac{\partial C}{\partial S} \cdot \Delta S + \frac{\partial C}{\partial t} \cdot \Delta t + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} (\Delta S)^2 + o(\Delta t),$$

where (as we have already seen in the demonstration of Theorem 6.1) $\Delta S$ is of size $\sqrt{\Delta t}$ and the terms summarized in $o(\Delta t)$ are of size smaller than $\Delta t$. Neglecting all terms but the first, which is of size $\sqrt{\Delta t}$, the approximation used in delta hedging is obtained:

$$\Delta C \approx \Delta \cdot \Delta S.$$

Also taking the terms of size $\Delta t$ into account it follows that:

$$\Delta C \approx \Delta \cdot \Delta S + \Theta \cdot \Delta t + \frac{1}{2}\Gamma(\Delta S)^2,$$

where $\Theta = \partial C/\partial t$ is the option's *theta* and $\Gamma = \partial^2 C/\partial S^2$ is the option's *gamma*. $\Theta$ is also called the option's *time decay*. For a call option on a non-dividend paying stock it follows from the Black–Scholes formula (6.24):

$$\Theta = -\frac{\sigma S}{2\sqrt{\tau}}\,\varphi(y + \sigma\sqrt{\tau}) - rKe^{-r\tau}\Phi(y) \tag{6.31}$$

$$\Gamma = \frac{1}{\sigma S\sqrt{\tau}}\,\varphi(y + \sigma\sqrt{\tau}), \tag{6.32}$$

where $y$ is defined in equation (6.25).

Figures 6.9 and 6.10 display theta and gamma given by equation (6.32) respectively (6.32) as a function of stock price and time to maturity. Most sensitive to movements in stock prices are at–the–money options with a short time to maturity. Consequently, to hedge such options the portfolio has to be rebalanced frequently.

Assuming a delta neutral portfolio *gamma hedging* consists of buying or selling further derivatives to achieve a gamma neutral portfolio, i.e. $\Gamma = 0$, and thereby making the portfolio value even more insensitive to changes in the stock price. Note that on the one hand neither stocks nor future contracts can be used for gamma hedging strategies since both have a constant $\Delta$ and thus a zero gamma $\Gamma = 0$. On the other hand, however, those instruments can be used to make a gamma neutral portfolio delta neutral without affecting the portfolio's gamma neutrality. Consider an option position with a gamma of $\Gamma$. Using $w$ contracts of an option traded on a stock exchange with a gamma of $\Gamma_B$, the portfolio's gamma is $\Gamma + w\Gamma_B$. By setting $w = -\Gamma/\Gamma_B$ the resulting gamma for the portfolio is 0.

**Example 6.9**
*Let a portfolio of* USD *options and US–Dollars be delta neutral with a gamma of* $\Gamma = -150\,000$. *A* USD*–call trade on the exchange with* $\Delta_B = 0.52$ *and*

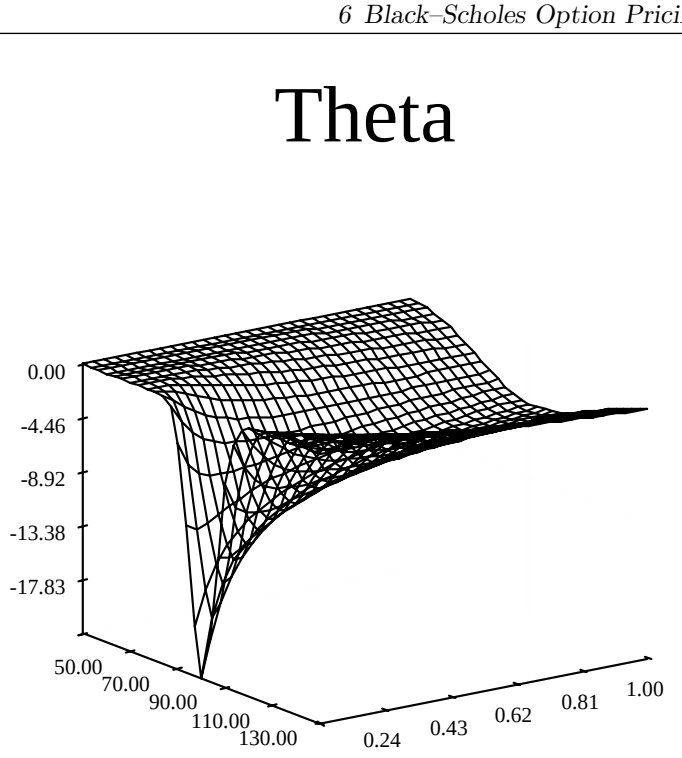



Figure 6.9: Theta as a function of stock price (right axis) and time to maturity (left axis). SFEtheta

$\Gamma_B = 1.20$. *By adding* $-\Gamma/\Gamma_B = 125\ 000$ *contracts of this option the portfolio becomes gamma neutral. Unfortunately, its delta will be* $125\ 000 \cdot \Delta_B = 65\ 000$. *The delta neutrality can be achieved by selling 65 000* USD *without changing the gamma.*

Contrary to the evolution of the stock price the expiry of time is deterministic, and time does not involve any risk increasing the randomness. If both $\Delta$ and $\Gamma$ are 0 then the option value changes (approximately risk free) at a rate $\Theta = \Delta C/\Delta t$. The parameter $\Theta$ is for most options negative, i.e. the option value decreases as the maturity date approaches.

From Black–Scholes's formula (6.24) it follows for a delta neutral portfolio consisting of stock options

$$rV = \Theta + \frac{1}{2}\sigma^2 S^2 \Gamma,$$

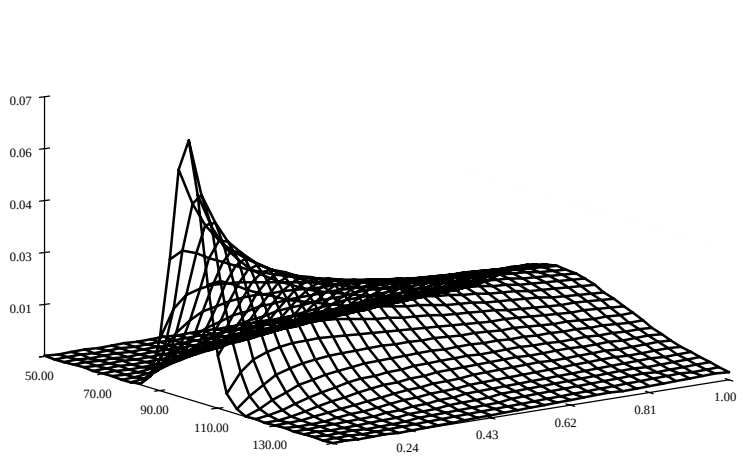



Figure 6.10: Gamma as a function of stock price (right axis) and time to maturity (left axis). SFEgamma

with $V$ denoting the portfolio value. $\Theta$ and $\Gamma$ depend on each other in a straightforward way. Consequently, $\Theta$ can be used instead of $\Gamma$ to gamma hedge a delta neutral portfolio.

### 6.4.3 Rho and Vega

Black–Scholes' approach proceeds from the assumption of a constant volatility $\sigma$. The appearance of smiles indicates that this assumption does not hold in practice. Therefore, it can be useful to make the portfolio value insensitive to changes in volatility. By doing this, the *vega* of a portfolio (in literature sometimes also called lambda or kappa) is used, which for a call option is defined by $\mathcal{V} = \frac{\partial C}{\partial \sigma}$.

For stocks and future contracts it holds $\mathcal{V} = 0$. Thus, in order to set up a vega hedge one has to make use of traded options. Since a vega neutral portfolio is not necessarily delta neutral two distinct options have to be involved to achieve simultaneously $\mathcal{V} = 0$ and $\Gamma = 0$.

From Black–Scholes' formula (6.24) and the variable $y$ defined in equation (6.25) it follows that the vega of a call option on a non dividend paying stock is given by:

$$\mathcal{V} = S\sqrt{\tau}\varphi(y + \sigma\sqrt{\tau}). \tag{6.33}$$

Since the Black–Scholes formula was derived under the assumption of a constant volatility it is actually not justified to compute the derivative of (6.24) with respect to $\sigma$. However, the above formula for $\mathcal{V}$ is quite similar to an equation for $\mathcal{V}$ following on from a more general stochastic volatility model. For that reason, equation (6.33) can be used as an approximation to the real vega.

Figure 6.11 displays the vega given by equation (6.33) as a function of stock price and time to maturity. When the option is in–the–money, vega is low. If the stock price approaches the strike price, vega increases and reaches its peak when the option becomes at–the–money. Further, if the option becomes out–of–the–money, the sensitivity of the option with respect to volatility is low again. Thus, at–the–money options with a long time to maturity are most sensitive to changes in volatility.

In order to protect the portfolio against implied volatility fluctuations, investors can use a *vega hedging* strategy. If the vega of a portfolio and of an option are denoted by $\mathcal{V}_{Port}$ and $\mathcal{V}_{Opt}$ respectively, one can take a $-\frac{\mathcal{V}_{Port}}{\mathcal{V}_{Opt}}$ position in the option to make a portfolio vega neutral. However, a vega neutral position is not stable and thus, even small changes in stock prices might cost investors a lot of money.

Finally, the call option's risk associated with movements in interest rates can be reduced by using *rho* to hedge the position:

$$\rho = \frac{\partial C}{\partial r}.$$

For a call on a non dividend paying stock it follows from equation (6.24)

$$\rho = K\ \tau\ e^{-r\tau}\Phi(y).$$

When hedging currency options domestic as well as foreign interest rates have to be taken into account. Consequently, rho hedging strategies need to consider two distinct values $\rho_1$ and $\rho_2$.

### 6.4.4 Volga and Vanna

*Volga*, also known as volgamma or vomma, is the sensitivity of vega to the change in implied volatility. Hence, it can also be defined as the second

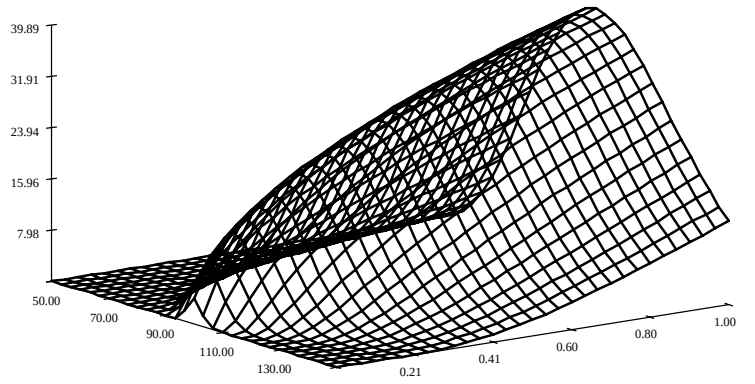



Figure 6.11: Vega as a function of stock price (right axis) and time to maturity (left axis). SFEvega

derivative of the option price with respect to volatility:

$$Volga = \frac{\partial \mathcal{V}}{\partial \sigma} = \frac{\partial^2 C}{\partial \sigma^2}.$$

Calculating volga from the Black-Scholes formula, for a call option we obtain:

$$Volga = S \cdot \sqrt{\tau} \cdot \frac{y \cdot \left(y + \sigma\sqrt{t}\right)}{\sigma} \cdot \varphi\left(y + \sigma\sqrt{t}\right) \tag{6.34}$$

Figure 6.12 displays the vanna given by equation (6.34) as a function of stock price and time to maturity.

If the option approaches at–the–money, volga becomes small, i.e., vega changes slowly. Consequently, the adjustments to keep a portfolio vega neutral need to be made relatively infrequently. On the contrary, if the option approaches in–the–money or out–of–the–money, volga becomes high, i.e., if stock price approaches strike price, the behaviour of vega is unstable.

The sensitivity of vega with respect to the stock price is given by *vanna*:

$$Vanna = \frac{\partial \mathcal{V}}{\partial S} = \frac{\partial^2 C}{\partial \sigma \partial S}.$$

# Volga

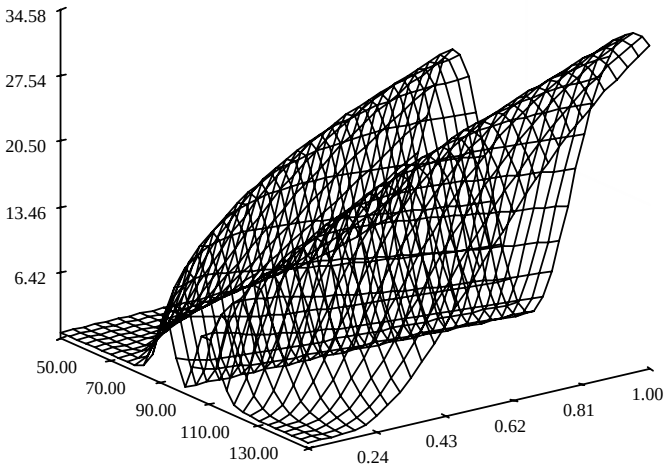



Figure 6.12: Volga as a function of stock price (right axis) and time to maturity (left axis). SFEvolga

Vanna is derived from the Black-Scholes formula for a call option and takes the following form:

$$Vanna = \left(\sqrt{\tau + \frac{1}{\sigma}}\right) \cdot \varphi\left(y + \sigma\sqrt{\tau}\right) \tag{6.35}$$

Figure 6.13 displays the vanna given by equation (6.35) as a function of stock price and time to maturity.

### 6.4.5 Historical and Implied Volatility

A property of the Black–Scholes formulae (6.22), (6.24) is that all option parameters are known, except the volatility parameter $\sigma$. In practical appli-

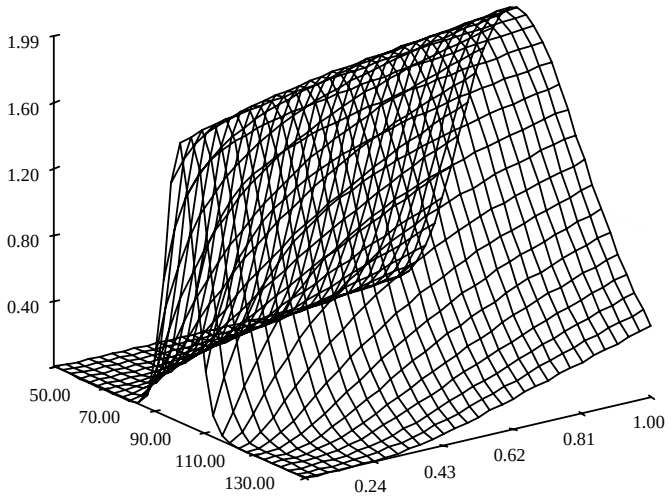



Figure 6.13: Vanna as a function of stock price (right axis) and time to maturity (left axis). SFEvanna

cations $\sigma$ is estimated from available stock price observations or from prices of similar products traded on an exchange.

*Historical volatility* is an estimator for $\sigma$ based on the variability of the underlying stock in the past. Let $S_0, \ldots, S_n$ be the stock prices at times $0, \Delta t, 2\Delta t, \ldots, n\Delta t$. If the stock price $S_t$ is modelled as Brownian motion, the logarithmic relative increments

$$R_t = \log \frac{S_t}{S_{t-1}}, \quad t = 1, \ldots, n$$

are independent and identical normally distributed random variables. $R_t$ is the increment $Y_t - Y_{t-1}$ of the logarithmic stock price $Y_t = \log S_t$ which as we saw in Section 5.4 is in a small time interval of length $\Delta t$ a Wiener process

with variance $\sigma^2$. Consequently the variance of $R_t$ is given by

$$v = \mathrm{Var}(R_t) = \sigma^2 \cdot \Delta t.$$

A good estimator for $\mathrm{Var}(R_t)$ is the sample variance

$$\hat{v} = \frac{1}{n-1} \sum_{t=1}^{n} (R_t - \bar{R}_n)^2$$

with $\bar{R}_n = \frac{1}{n} \sum_{t=1}^{n} R_t$ being the sample average. $\hat{v}$ is unbiased, i.e. $\mathsf{E}[\hat{v}] = v$, and the random variable

$$(n-1)\,\frac{\hat{v}}{v}$$

is $\chi^2_{n-1}$ distributed (chi-square distribution with $n-1$ degrees of freedom). In particular this implies that the mean squared relative estimation error of $\hat{v}$ is given by

$$\mathsf{E}\left(\frac{\hat{v}-v}{v}\right)^2 = \frac{1}{(n-1)^2}\ \mathrm{Var}\left((n-1)\frac{\hat{v}}{v}\right) = \frac{2}{n-1}.$$

Since it holds $v = \sigma^2 \Delta t$ an estimator for the volatility $\sigma$ based on historical stock prices is

$$\hat{\sigma} = \sqrt{\hat{v}/\Delta t}.$$

By means of a Taylor expansion of the square root function and by means of the known quantities $\mathsf{E}[\hat{v}]$ and $\mathrm{Var}(\hat{v}/v)$ it follows that $\hat{\sigma}$ is unbiased neglecting terms of size $n^{-1}$ :

$$\mathsf{E}\,\hat{\sigma} = \sigma + \mathcal{O}\left(\frac{1}{n}\right)\ ,$$

and that the mean squared relative estimation error of $\hat{\sigma}$ is given by

$$\mathsf{E}\left(\frac{\hat{\sigma}-\sigma}{\sigma}\right)^2 = \frac{1}{2(n-1)} + \mathcal{O}\left(\frac{1}{n}\right)\ ,$$

again neglecting terms of size smaller than $n^{-1}$. Thanks to this relationship the reliability of the estimator $\hat{\sigma}$ can be evaluated.

**Sample parameter selection:**

a) As data daily settlement prices $S_0, \ldots, S_n$ are often used. Since $\sigma$ is in general expressed as an annualized volatility $\Delta t$ corresponds to one day on a yearly basis. Working with calender day count convention $\Delta t = \frac{1}{365}$. Unfortunately, no data is available for weekends and holidays. The following empirical argument favours ignoring weekends and holidays:

If the stock dynamics behaved on Saturdays and Sundays as they do on trading days, even if the dynamics were not observed, then standard deviation of the change in the stock price from Friday to Monday would be three times as large as the standard deviation between two trading days, say Monday to Tuesday. This follows on from the fact that for the Wiener process $Y_t = \log S_t$ the standard deviation of the increment $Y_{t+\delta} - Y_t$ is $\sigma \cdot \delta$. Empirical studies of stock markets show, however, that both standard deviations are proportional, with a constant of around 1 but in any case significantly smaller than 3. Put in other words, the volatility decreases at the weekend. A conclusion is that trading increases volatility, and that the stock variability is not solely driven by external economic influences. Estimating volatility should therefore be done by exclusively considering trading days. Usually a year is supposed to have 252 trading days, i.e. $\Delta t = \frac{1}{252}$.

Concerning monthly data, $\Delta t = \frac{1}{12}$ is applied. In Section 3.3 we calculated an annual volatility of 19% based on the monthly DAX data.

SFEsumm

b) Theoretically, the larger $n$ the more reliable $\hat{\sigma}$. However, empirically the volatility is not constant over longer periods of time. That is to say that stock prices from the recent past contain more information about the current $\sigma$ as do stock prices from long ago. As a compromise the closing prices of the last 90 days or 180 days are used respectively. Some authors advise the use of historical data of a period which has the same length as the period in the future to which the estimated volatility will be applied. In other words, if you want to compute the value of a call expiring in 9 months you should use the closing prices of the past 9 months.

The *implied volatility* of an option is computed from its market price observed on an exchange and not from the prices of the underlying as is case for the historical volatility. Consider a European call on a non-dividend paying stock ($d = 0,\ b = r$), which has a quoted market price of $C_B$, then its implied volatility $\sigma_I$ is given by solving

$$S\ \Phi(y + \sigma_I\sqrt{\tau}) - e^{-r\tau} K\ \Phi(y) = C_B$$

$$\text{with} \qquad y = \frac{1}{\sigma_I\sqrt{\tau}} \{\log \frac{S}{K} + (r - \frac{1}{2}\sigma_I^2)\tau\}.$$

$\sigma_I$ is the value of the volatility which, if substituted into the Black-Scholes formula (6.24), would give a price equal to the observed market price $C_B$. $\sigma_I$ is implicitly defined as a solution to the above equation, and has to be com-

puted numerically due to the fact that the Black–Scholes formula cannot be inverted.

The implied volatility can be used to get an idea of the market view of the stock volatility. It is possible to construct an estimator using implied volatilities of options on the same stock but which are different in time to maturity $\tau$ and exercise price $K$. A weighting scheme takes the option price dependence on the volatility into account.

SFEVolSurfPlot

**Example 6.10**
*Consider two traded options on the same underlying. One is at–the-money (ATM) and the other is deep ITM with volatilities of* $\sigma_{I1} = 0.25$ *respectively* $\sigma_{I2} = 0.21$. *At–the–money the dependence of option price and volatility is particulary strong. That is, the price of the first option contains more information about the stock volatility and* $\sigma_{I1}$ *can be considered a more reliable volatility estimate. Thus the estimator combining both implied volatilities should attribute a higher weight to* $\sigma_{I1}$, *as for example*

$$\tilde{\sigma} = 0.8 \cdot \sigma_{I1} + 0.2 \cdot \sigma_{I2}.$$

*Some authors suggest setting* $\tilde{\sigma} = \sigma_{Im}$ *with* $\sigma_{Im}$ *being the volatility of the option which is most sensitive to changes in* $\sigma$, *i.e. the option with the highest vega* $\partial C/\partial\sigma$ *in absolute terms.*

### 6.4.6 Realised Volatility

*Realised volatility* is constructed from high-frequency intra-day returns. Most stochastic volatility models, do not include high-frequency intra-day data for the calculation of daily volatility, and are inadequate in reflecting reality. Realised volatility, on the contrary, takes into consideration high-frequency intraday data for forecasting daily and lower frequency volatility and the distribution of returns. Based on the theory of quadratic variation, realised volatility is given by the sum of all intra-period high-frequency squared returns, period by period.

Denote as above $Y_t = \log S_t$ the logarithmic stock price and

$$R_t = Y(t\Delta) - Y\left\{(t-1)\Delta\right\}, \quad t = 1, 2, ..., n$$

the returns over an interval of length $\Delta$, e.g. 1 day. Let $\sigma^2(t)$ denote the variance of $R_t$. We split the time interval $[(t-1)\Delta, t\Delta]$ into $M$ small

sub-intervals of length $\Delta/M$ each, corresponding to intra-day observations during the day $t$. Then, the realised volatility $\{Y\}_2^{1/2}$ is modelled via the sum of squared intra-day changes over a day:

$$\{Y\}_t = \sum_{j=1}^{M} \left[ Y\left\{(t-1)\Delta + \frac{\Delta j}{M}\right\} - Y\left\{(t-1)\Delta + \frac{\Delta(j-1)}{M}\right\}\right]^2,$$

$\{Y\}_2^{1/2}$ is an estimate of the *actual volatility* $\sigma(t) = \{\mathrm{Var}(R_t)\}^{1/2}$. For example, for a 24-hour market, daily realised volatility based on 5minute underlying returns is defined as the sum of 288 intra-day squared 5minute returns, taken day by day.

## 6.5 Recommended Literature

The classic papers of Black and Scholes (1973) and Merton (1973) which established modern derivatives pricing theory are worth reading. As does this book, Wilmott, Howison and Dewynne (1995) present an extensive introduction to mathematics of financial derivatives without martingale theory. Two influential works contributing to modern financial mathematics but which apply more advanced results of the theory of stochastic processes are Harrison and Pliska (1981) and Delbaen and Schachermayer (1994). A discussion of the mathematical foundations of absence of arbitrage is given by Jensen and Nielsen (1996). Korn and Korn (1999) and Korn and Korn (2001) provide a compact introduction to modern financial mathematics. For the advanced mathematician Duffie (1996) and Baxter and Rennie (1996) represent good starts into derivative pricing using martingale theory. Korn (1999) puts the focus on problems arising in hedging and portfolio optimisation. Crack (2004) gives a clear explanation of Black-Scholes option pricing theory, by discussing direct applications of the theory to trading, and the differences between the theoretical Black-Scholes world and the real world. Hull (2006) derives the Black-Scholes model for valuing the options on a non-dividend paying stock and shows how the model can be extended to deal with options on dividend paying stocks. The explanation of how volatility can be either estimated from the historical data or implied from option prices is also given by Hull (2006).

# 7 Binomial Model for European Options

A large range of options exist for which the boundary conditions of the Black-Scholes differential equation are too complex to solve analytically; an example being the American option. One therefore has to rely on numerical price computation. The best known methods for this approximate the stock price process by a discrete time stochastic process, or, as in the approach followed by Cox, Ross, Rubinstein, model the stock price process as a discrete time process from the start. By doing this, the options time to maturity $T$ is decomposed into $n$ equidistant time steps of length

$$\Delta t = \frac{T}{n}.$$

We consider therefore the discrete time points

$$t_j = j\Delta t,\ j = 0, ..., n.$$

By $S_j$ we denote the stock price at time $t_j$. At the same time, we discrete the set of values the stock price can take, such that it takes on finite many values $S_j^k, k = 1, ..., m_j$, with $j$ denoting the point of time and $k$ representing the value. If the stock price is in time $t_j$ equal to $S_j^k$, then it can jump in the next time step to one of $m_{j+1}$ new states $S_{j+1}^l,\ l = 1, ..., m_{j+1}$. The probabilities associated to these movements are denoted by $p_{kl}^j$:

$$p_{kl}^j = \mathrm{P}(S_{j+1} = S_{j+1}^l | S_j = S_j^k),$$

with

$$\sum_{l=1}^{m_{j+1}} p_{kl}^j = 1, \quad 0 \leq p_{kl}^j \leq 1.$$

If we know the stock price at the current time, we can build up a tree of possible prices up to a certain point in time, for example the maturity date $T = t_n$. Such a tree is also called *stock price tree*. Should the option price be known at the final point in time $t_n$ of the stock price tree, for example by means of the options intrinsic value, the option value at time $t_{n-1}$ can

be computed (according to (6.24)) as the discounted conditional expectation of the corresponding option prices at time $t_n$ given the stock price at time $t_{n-1}$ :

$$\begin{aligned} V(S_{n-1}^k, t_{n-1}) &= e^{-r\Delta t}\mathsf{E}\{V(S_n, t_n) \,|\, S_{n-1} = S_{n-1}^k\} \\ &= e^{-r\Delta t}\sum_{l=1}^{m_n} p_{kl}^{n-1} V(S_n^l, t_n). \end{aligned} \tag{7.1}$$

$V(S,t)$ again denotes the option value at time $t$ if the underlying has a price of $S$. Repeating this step for the remaining time steps $t_j,\ j = n-2, n-3, ..., 0$, means that the option prices up to time $t = 0$ can be approximated.

## 7.1 Cox–Ross–Rubinstein Approach to Option Pricing

As the simplest example to price an option, we consider the approach by Cox, Ross and Rubinstein (CRR) which is based on the assumption of a *binomial model*, and which can be interpreted as a numerical method to solve the Black–Scholes equation. We will look at European options exclusively and assume, for the time being, that the underlying pays no dividends within the time to maturity. Again, we discretize time and solely consider the points in time $t_0 = 0, t_1 = \Delta t, t_2 = 2\Delta t, ..., t_n = n\Delta t = T$ with $\Delta t = \frac{T}{n}$. The binomial model proceeds from the assumption that the discrete time stock price process $S_j$ follows a geometric random walk (see Chapter 4), which is the discrete analogue of the geometric Brownian motion. The binomial model has the special feature that at any point in time the stock price has only two possibilities to move:

- either the price moves at rate $u$ and with probability $p$ in one direction (for example it moves $u$p)
- or the price moves at rate $d$ and with probability $1 - p$ in another direction (for example it moves $d$own).

Using the notation introduced above, if the stock price in time $t_j$ is equal to $S_j^k$ then in time $t_{j+1}$ it can take only the values $u \cdot S_j^k$ and $d \cdot S_j^k$. The probabilities $p$ and $q$ are independent of $j$. All other probabilities $p_{kl}^j$ associated to $S_{j+1}^l \neq u \cdot S_j^k$ and $\neq d \cdot S_j^k$ are 0.

In order to approximate the Black–Scholes differential equation by means of the Cox–Ross–Rubinstein approach, the probabilities $p, q$ as well as the rates $u, d$ have to be chosen such that in the limit $\Delta t \to 0$ the binomial model

converges to a geometric Brownian motion. That is, arguing as in (6.22) the conditional distribution of $\ln S_{j+1}$ given $S_j$ must be asympotically a normal distribution with expectation parameter $\ln S_j + (b - \frac{1}{2}\sigma^2)\Delta t$ and variance parameter $\sigma^2 \Delta t$. However, the conditional distribution of $\ln S_{j+1}$ given $S_j$ implied by the binomial model is determined by $\ln(u \cdot S_j), \ln(d \cdot S_j)$ and their associated probabilities $p$ and $q$. We set the parameters of the geometric random walk such that the conditional expectations and variances implied by the binomial model are equal to their asymptotic values for $\Delta t \to 0$. Taking into account that $p + q = 1$ we obtain three equations for the four unknown variables $p, q, u$ and $d$ :

$$\begin{aligned} p + q &= 1, \\ E \stackrel{\text{def}}{=} p \ln(u \cdot S_j) + q \ln(d \cdot S_j) &= \ln(S_j) + (b - \frac{1}{2}\sigma^2)\Delta t, \\ p\{\ln(u \cdot S_j) - E\}^2 + q\{\ln(d \cdot S_j) - E\}^2 &= \sigma^2 \Delta t. \end{aligned}$$

Due to the first equation, the current stock price $S_j$ disappears from the remaining equations. By substituting $q = 1 - p$ into the latter two equations, we obtain, after some rearrangements, two equations and three unknown variables:

$$\begin{aligned} p \ln(\frac{u}{d}) + \ln d &= (b - \frac{1}{2}\sigma^2)\Delta t, \\ (1-p)p\{\ln\left(\frac{u}{d}\right)\}^2 &= \sigma^2 \Delta t. \end{aligned}$$

To solve this nonlinear system of equations we introduce a further condition

$$u \cdot d = 1,$$

i.e. if the stock price moves up and subsequently down, or down and subsequently up, then it takes its initial value two steps later. This recombining feature is more than only intuitively appealing. It simplifies the price tree significantly. At time $t_j$ there are only $m_j = j + 1$ possible values the stock price $S_j$ can assume. More precisely, given the starting value $S_0$ at time $t_0$ the set of possible prices at time $t_j$ is

$$S_j^k = S_0 u^k d^{j-k}, \; k = 0, ..., j,$$

because it holds $S_{j+1}^{k+1} = u \cdot S_j^k$ and $S_{j+1}^k = S_j^k / u$. In the general case there would be $m_j = 2^j$ possible states since then not only the number of up and down movements would determine the final state but also the order of the up and down movements.

Solving the system of three equations for $p, u, d$ and neglecting terms being small compared to $\Delta t$ it holds approximatively:

$$p = \frac{1}{2} + \frac{1}{2}(b - \frac{1}{2}\sigma^2)\frac{\sqrt{\Delta t}}{\sigma}, \quad u = e^{\sigma\sqrt{\Delta t}}, \quad d = \frac{1}{u}. \tag{7.2}$$

For the option price at time $t_j$ and a stock price $S_j = S_j^k$ we use the abbreviation $V_j^k = V(S_j^k, t_j)$. As in equation (7.1) we obtain the option price at time $t_j$ by discounting the conditional expectation of the option price at time $t_{j+1}$ :

$$V_j^k = e^{-r\Delta t}\{pV_{j+1}^{k+1} + (1-p)V_{j+1}^k\}. \tag{7.3}$$

At maturity $T = t_n$ the option price is known. In case of a European option we have

$$V_n^k = \max\{0, S_n^k - K\},\ k = 0, ..., n. \tag{7.4}$$

Beginning with equation (7.1) and applying equation (7.3) recursively all option values $V_j^k$, $k = 0, ..., j$, $j = n-1, n-2, ..., 0$ can be determined.

**Example 7.1**
*An example of a call option is given in Table 7.1. First the tree of stock prices is computed. Since* $\Delta t = \tau/n = 0.1$ *it follows from equation (7.2) that* $u = 1.0823$. *Given the current stock price* $S_0 = 230$ *the stock can either increase to* $S_1^1 = uS_0 = 248.92$ *or decrease to* $S_1^0 = S_0/u = 212.52$ *after the first time step. After the second time step, proceeding from state* $S_1 = S_1^1$ *the stock price can take the values* $S_2^2 = uS_1 = 269.40$ *or* $S_2^1 = S_1/u = 230$, *proceeding from* $S_1 = S_1^0$ *it can move to* $S_2^1 = 230$ *or* $S_2^0 = 196.36$ *and so on. At maturity, after 5 time steps, the stock price* $S_5$ *can take the following six values* $S_5^5 = u^5 S_0 = 341.51, S_5^4 = u^3 S_0 = 291.56, ..., S_5^0 = S_0/u^5 = 154.90$.

*Following, given the tree of stock prices, we compute the option price at maturity applying equation (7.4), for example* $V_5^4 = V(S_5^4, t_5) = S_5^4 - K = 81.561$ *or* $V_5^1 = 0$, *since* $S_5^1 = 181.44 < K$. *Equation (7.2) implies* $p = 0.50898$, *since the cost of carry b are equal to the risk free interest rate r when no dividends are paid. Proceeding from the option's intrinsic values at maturity we compute recursively the option values at preceding points of time by means of equation (7.3). With* $V_5^4 = 81.561, V_5^3 = 38.921$ *we obtain the option value* $V_4^3 = 60.349$ *at time* $t_4 = 0.4$ *corresponding to a stock price* $S_4 = S_4^3 = 269.40$ *by substituting the known values of* $p, r, \Delta t$. *Analogously we obtain the option value* $V_0^0 = 30.378$ *at time* $t_0 = 0$ *and current stock price* $S_0 = 230$ *by means of equation (7.3) and the time* $t_1 = 0.1$ *option values* $V_1^1 = 44.328, V_1^0 = 16.200$.

*Using only 5 time steps* 30.378 *is just a rough approximation to the theoretical call value. However, comparing prices implied by the Black–Scholes formula*

*(6.24) to prices implied by the Cox–Ross–Rubinstein approach for different time steps n the convergence of the numerical binomial model solution to the Black–Scholes solution for increasing n is evident (see Table 7.2).*

| | |
|---|---|
| Current stock price $S_t$ | 230.00 |
| Exercise price $K$ | 210.00 |
| Time to maturity $\tau$ | 0.50 |
| Volatility $\sigma$ | 0.25 |
| Risk free rate $r$ | 0.04545 |
| Dividend | none |
| Time steps | 5 |
| Option type | European call |

| Stock prices | Option prices | | | | | |
|---|---|---|---|---|---|---|
| 341.50558 | | | | | | 131.506 |
| 315.54682 | | | | | 106.497 | |
| 291.56126 | | | | 83.457 | | 81.561 |
| 269.39890 | | | 62.237 | | 60.349 | |
| 248.92117 | | 44.328 | | 40.818 | | 38.921 |
| 230.00000 | 30.378 | | 26.175 | | 20.951 | |
| 212.51708 | | 16.200 | | 11.238 | | 2.517 |
| 196.36309 | | | 6.010 | | 1.275 | |
| 181.43700 | | | | 0.646 | | 0.000 |
| 167.64549 | | | | | 0.000 | |
| 154.90230 | | | | | | 0.000 |
| Time | 0.00 | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 |

Table 7.1: Evolution of option prices (no dividend paying underlying)

SFEBiTree

The numerical procedure to price an option described above does not change if the underlying pays a continuous dividend at rate $d$. It is sufficient to set $b = r - d$ instead of $b = r$ for the cost of carry. Dividends paid at discrete points of time, however, require substantial modifications in the recursive option price computation; these will be discussed in the following section.

**Example 7.2**
*We consider a call on US–Dollar with a time to maturity of* 4 *months, i.e.* $\tau = 1/3$ *years, a current exchange rate of* $S = 1.50$ EUR/USD *and an exercise price* $K = 1.50$ EUR/USD. *The continuous dividend yield, which corresponds to the US interest rate, is assumed to be* 1%, *and the domestic interest rate is*

| Time steps | 5 | 10 | 20 | 50 | 100 | 150 | Black-Scholes |
|---|---|---|---|---|---|---|---|
| Option value | 30.378 | 30.817 | 30.724 | 30.751 | 30.769 | 30.740 | 30.741 |

Table 7.2: Convergence of the price implied by the binomial model to the price implied by the Black–Scholes formula

9%. *It follows that the cost of carry being the difference between the domestic and the foreign interest rate is equal to* $b = r - d = 8\%$. *Table 7.3 gives as in the previous example the option prices implied by the binomial model.*

| | |
|---|---|
| Current EUR/USD–price $S_t$ | 1.50 |
| Exercise price $K$ | 1.50 |
| Time to maturity $\tau$ | 0.33 |
| Volatility $\sigma$ | 0.20 |
| Risk free interest rate $r$ | 0.09 |
| Continuous dividend $d$ | 0.01 |
| Time steps | 6 |
| Option type | European call |

| Price | Option prices | | | | | | |
|---|---|---|---|---|---|---|---|
| 1.99034 | | | | | | | 0.490 |
| 1.89869 | | | | | | 0.405 | |
| 1.81127 | | | | | 0.324 | | 0.311 |
| 1.72786 | | | | 0.247 | | 0.234 | |
| 1.64830 | | | 0.180 | | 0.161 | | 0.148 |
| 1.57240 | | 0.127 | | 0.105 | | 0.079 | |
| 1.50000 | 0.087 | | 0.067 | | 0.042 | | 0.000 |
| 1.43093 | | 0.041 | | 0.022 | | 0.000 | |
| 1.36504 | | | 0.012 | | 0.000 | | 0.000 |
| 1.30219 | | | | 0.000 | | 0.000 | |
| 1.24223 | | | | | 0.000 | | 0.000 |
| 1.18503 | | | | | | 0.000 | |
| 1.13046 | | | | | | | 0.000 |
| Time | 0.00 | 0.06 | 0.11 | 0.17 | 0.22 | 0.28 | 0.33 |

Table 7.3: Evolution of option prices (with continuous dividends)

SFEBiTree

## 7.2 Discrete Dividends

In case where dividend payments are made at discrete points in time, the tree of stock prices changes. By changing the price tree we have to distinguish two different cases. In the first case, dividends are paid as a percentage of

the stock price. In the second case, dividends are paid as a fixed amount of money. We confine ourselves to the case that dividends are paid only once during the time to maturity, say, at time $t^*, 0 < t^* \leq T$. Dividends paid at several points of time can be dealt with analogously. We assume that the underlying is a stock.

Using no arbitrage arguments it can be shown that the stock price jumps down by the amount of the dividend at the time the dividend is paid. Let us consider the following argument to visualise this. At time $t^* - dt$, which is immediately before the dividend is paid, we buy the stock, cash in the dividend, and sell the stock at time $t^* + dt$. By doing this, we make a gain of $D + S_{t^*+dt} - S_{t^*-dt}$, which for $dt \to 0$ would be without tisk and, therefore, has to be zero if arbitrage is excluded. This is guaranteed if $S_t$ jumps down by $D$ at time $t^*$.

### 7.2.1 Dividends as a Percentage of the Stock Price

Suppose that $t^*$ is contained, say, in the $i$th time interval, i.e. $t_{i-1} < t^* \leq t_i$. Let the dividend paid at time $t_i$ be a percentage $\delta$ of the stock price, that is the dividend amount that is paid is equal to $\delta S_i$. It follows that the stock price at time $t_i$ is smaller by the dividend amount than the stock price without the dividend payment. Accordingly, all stock prices in the tree after time $t_i$ change in the same way: all prices $S_j^k$, $j \geq i$, are multiplied by the factor $(1 - \delta)$. Following this correction the option values can be determined recursively as in the no dividend case.

**Example 7.3**
*We consider a call option on a stock paying a dividend of $\delta = 1\%$ of the stock price at time* 0.15. *All other parameters of this example are those already given in Table 7.1. The results are shown in Table 7.4. First we ignore the dividend and compute the stock price tree as shown in Table 7.1. Following, all stock prices from the dividend date on, i.e. from time $t_2 = 0.2$ on (note that we have divided the time period into 5 time steps $0 \leq t \leq 0.5$), are multiplied by the factor $(1 - \delta)$. In Table 7.4 the values in parentheses correspond to the stock prices that are decreased by the dividend amount, i.e. $S_j^k$, $j < i = 2$ respectively $0.99 \cdot S_j^k$, $j \geq i = 2$. Thus, the option prices at maturity change due to equation (7.4), for example $V_5^4 = V(0.99 \cdot S_5^4, t_5) = 0.99 \cdot 291.56 - K = 78.646$. Having determined the option values at maturity the preceding option values are again computed by recursively applying equation (7.2). Note, $V_j^k$ corresponds to the stock price $0.99 \cdot S_j^k$ rather than to $S_j^k$, for $j \geq 2$, i.e. $t_j \geq t^*$. However, the current time $t_0 = 0 < t^* = 0.15$ is*

*not concerned, i.e.* $V_0^0 = 28.384$ *is still the option price corresponding to the current stock price* $S_0 = 230$.

| | |
|---|---|
| Current stock price $S_t$ | 230.00 |
| Exercise price $K$ | 210.00 |
| Time to maturity $\tau$ | 0.50 |
| Volatility $\sigma$ | 0.25 |
| Risk free interest rate $r$ | 0.04545 |
| Discrete dividend $\delta$ | 0.01 |
| Dividend date $t^*$ | 0.15 |
| Time steps | 5 |
| Option type | European call |

| Stock prices | Option prices | | | | | |
|---|---|---|---|---|---|---|
| 341.50558 | | | | | | 128.091 |
| 315.54682 | | | | | 103.341 | (338.09) |
| 291.56126 | | | | 80.542 | (312.39) | 78.646 |
| 269.39890 | | | 59.543 | (288.65) | 57.655 | (288.65) |
| 248.92117 | | 41.942 | (266.70) | 38.329 | (266.70) | 36.432 |
| 230.00000 | 28.384 | (248.92) | 24.087 | (246.43) | 18.651 | (246.43) |
| 212.51708 | (230.00) | 14.592 | (227.70) | 9.547 | (227.70) | 0.392 |
| 196.36309 | | (212.52) | 4.886 | (210.39) | 0.199 | (210.39) |
| 181.43700 | | | (194.40) | 0.101 | (194.40) | 0.000 |
| 167.64549 | | | | (179.62) | 0.000 | (179.62) |
| 154.90230 | | | | | (165.97) | 0.000 |
| | | | | | | (153.35) |
| Time | 0.00 | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 |
| Dividend | 1.00 | 1.00 | 0.99 | 0.99 | 0.99 | 0.99 |

Table 7.4: Evolution of option prices (dividends as a percentage of the stock price)

SFEBiTree

### 7.2.2 Dividends as a Fixed Amount of Money

We assume now that at an ex ante fixed point in time $t^*$ a fixed amount of money (for example 5.00 EUR) is paid. Now, the stock price jumps down by an amount which is independent of the stock price. It follows that the tree is not totally recombining anymore. The stock price tree splits up which can be visualized in a simple example. Suppose that at time $t^*$, $t_1 < t^* \leq t_2 < T$, a fixed dividend of $D$ is paid. Figure 7.1 shows the stock price tree for this example. Before the dividend payment at time $t_1$ the nodes correspond to stock prices of the kind $uS_0$ and $S_0/u$. After the dividend payment, however, stock prices at time $t_2$ are given by $u^2S_0 - D, S_0 - D$ and $S_0/u^2 - D$. Proceeding from these 3 prices the tree consists of 6 possible prices

in time $t_3$, at time $t_4$ it consists of 9 and so on. The stock price tree gets very vast the more time steps are considered, and is less useful for practical computations. To overcome this problem, we use the fact that the dividend is independent of the stock price and therefore not random anymore. We decompose the stock price $S_j$ in a random and a deterministic component:

$$S_j = \tilde{S}_j + D_j,$$

with $D_j$ being the current present value of the dividend payment, i.e. before dividend payment, it is the time $t_j \leq t^*$ discounted value of $D$, afterwards it is 0 :

$$D_j = \begin{cases} De^{-r(t^*-t_j)} & \text{, for } t_j \leq t^*, \\ 0 & \text{, for } t^* < t_j. \end{cases} \tag{7.5}$$

In particular, at maturity it holds $D_n = 0$ and $S_n = \tilde{S}_n$. In order to compute

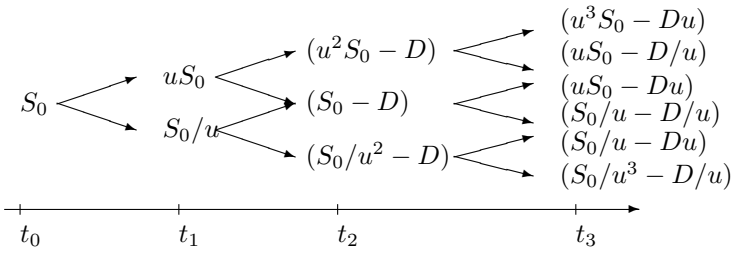



Figure 7.1: Evolution of the stock price tree (dividends as a fixed amount of money

the option price we first construct a stock price tree of the random stock price component $\tilde{S}_n$ beginning in $\tilde{S}_0 = S_0 - D_0$. Starting at maturity $T = t_n$ we obtain:

$$\tilde{V}_{n-1} = e^{-r\Delta t}\mathsf{E}[\max(0, \tilde{S}_n - K)|\tilde{S}_{n-1}]$$

The other option prices are given as in the no dividend case by:

$$\tilde{V}_{j-1} = e^{-r\Delta t}\mathsf{E}[\tilde{V}_j|\tilde{S}_{j-1}].$$

The original option prices then correspond to $\tilde{V}_j^k$ given above. However, they do not correspond to the stock price $\tilde{S}_j^k$, rather than to the actual stock price

$$S_j^k = \tilde{S}_j^k + D_j.$$

**Example 7.4**
*In this example, there are two dividend payments at time* $t_1^* = 0.25$ *and*

$t_2^* = 0.75$. *Both dividends are* $D^{(1)} = D^{(2)} = 1.00$ EUR. *The parameters and results are given in Table 7.5. First, we compute the time* $t_j$ *present value of all dividends with equation (7.5):* $D_j = D^{(1)}e^{-r(t_1^*-t_j)} + D^{(2)}e^{-r(t_2^*-t_j)}$ *for* $t_j \le t_1^*$, $D_j = D^{(2)}e^{-r(t_2^*-t_j)}$ *for* $t_1^* < t_j \le t_2^*$ *and* $D_j = 0$ *for* $t_2^* < t_j$. *In*

| | |
|---|---|
| Current stock price$S_t$ | 100.00 |
| Exercise price $K$ | 100.00 |
| Time to maturity $\tau$ | 1.00 |
| Volatility $\sigma$ | 0.30 |
| Risk free interest rate $r$ | 0.10 |
| Discrete dividend $D^{(1)}$ | 1.00 |
| Payment date $t_1^*$ | 0.25 |
| Discrete dividend $D^{(2)}$ | 1.00 |
| payment date $t_2^*$ | 0.75 |
| Time steps | 6 |
| Option type | European Put |

| Prices | Option prices | | | | | | |
|---|---|---|---|---|---|---|---|
| 204.55 | | | | | | | 0.000 |
| 180.97 | | | | | | 0.000 | (204.55) |
| 160.12 | | | | | 0.000 | (180.97) | 0.000 |
| 141.65 | | | | 0.179 | (161.10) | 0.000 | (160.11) |
| 125.32 | | | 1.373 | (142.63) | 0.394 | (141.65) | 0.000 |
| 110.88 | | 3.906 | (126.28) | 2.810 | (126.32) | 0.866 | (125.32) |
| 98.10 | 7.631 | (112.81) | 6.990 | (111.85) | 5.720 | (110.88) | 1.903 |
| 86.79 | (100) | 12.236 | (99.06) | 12.100 | (99.09) | 11.567 | (98.10) |
| 76.78 | | (88.72) | 18.775 | (87.76) | 19.953 | (86.79) | 23.215 |
| 67.93 | | | (77.74) | 27.211 | (77.78) | 30.421 | (76.78) |
| 60.10 | | | | (68.91) | 36.631 | (67.93) | 39.897 |
| 53.17 | | | | | (61.09) | 45.178 | (60.10) |
| 47.05 | | | | | | (53.17) | 52.955 |
| | | | | | | | (47.05) |
| Zeit | 0.00 | 0.17 | 0.33 | 0.50 | 0.67 | 0.83 | 1.00 |
| Div. $D_j$ | 1.903 | 1.935 | 0.960 | 0.975 | 0.992 | 0.00 | 0.00 |

Table 7.5: Evolution of option prices (discrete dividends as a fixed money amount)

SFEBiTree4

*particular, it holds that* $D_j = 0$ *for* $t_j > t_2^*$. *Below, we construct the stock price tree as in Table 7.1, but this time we start in* $\tilde{S}_0 = S_0 - D_0 = 98.10$ *rather than in* $S_0 = 100$. *Proceeding from the boundary values* $\tilde{V}_6^k = K - \tilde{S}_n^k, k = 0, ..., 3, \tilde{V}_6^k = 0, k = 4, ..., 6$ *we compute once again recursively the put prices at earlier points in time by means of equation (7.3). We have to take into account that, for example, the option price* $\tilde{V}_3^2 = 2.810$ *belongs to the stock price* $S_3^2 = \tilde{S}_3^2 + D_3 = 111.85$ *and not to* $\tilde{S}_3^2 = 110.88$, *which accounts for the dividend. It follows that the put option price at a current stock price* $S_0 = 100$ *is equal to* $\tilde{V}_0^0 = 7.631$.

## 7.3 Recommended Literature

The starting point to price options by means of binomial processes is the classic work of Cox, Ross and Rubinstein (1979) who introduce this approach as an independent method rather than only as a numeric approximation to the Black–Scholes equations. Baxter and Rennie (1996) provide a detailed and modern description of option pricing with binomial trees. The numerical aspects are extensively discussed by Dewynne, Howison and Wilmott (1993).

# 8 American Options

## 8.1 Arbitrage Relationship for American Options

It is complex to price American options since they can be exercised at any point in time up to the expiry date. The time the holder chooses to exercise the options depends on the spot price of the underlying asset $S_t$. In this sense the exercising time is a random variable itself. It is obvious that the Black-Scholes differential equations still hold as long as the options are not exercised. However the boundary conditions are so complicated that an analytical solution is not possible. In this section we study American options in more detail. The numerical procedures of pricing will also be discussed in the next section.

As shown in Section 2.1, the right to early exercise implies that the value of an American option can never drop below its intrinsic value. For example the value of an American put should not go below $\max(K - S_t, 0)$ with the exercise price $K$. In contrast this condition does not hold for European options. Thus American puts would be exercised before expiry date if the value of the option would drop below the intrinsic value.

Let's consider an American put on a stock with expiry date $T$. If the stock price $S_{t^*}$ at time $t^*$ is zero, then $S_t = 0$ holds for $t \geq t^*$ since the price process follows a geometric Brownian motion. It is then not worth waiting for a later exercise any more. If the put holder waits, he will lose the interest on the value $K$ that can be received from a bond investment for example. If $S_{t^*} = 0$, the value of the put at $t^*$ is $K$ which is the same as the intrinsic value. Since the respective European put cannot be exercised early, e.g. at time $t^*$, we can only get $K$ on the expiry date. If we discount it to time $t^*$ with $\tau^* = T - t^*$, we only get $Ke^{-r\tau^*}$ that is the value of the European put at time $t^*$. Obviously this value is smaller than the value of an American put and its intrinsic value. Figure 8.1 shows the put value with a continuous cost of carry $b$.

As we can see an early exercise of the put is probably necessary even before $S_t = 0$. For a certain critical stock price $S^{**}$, the loss of interest on the intrinsic value, which the holder can receive by exercising it immediately,

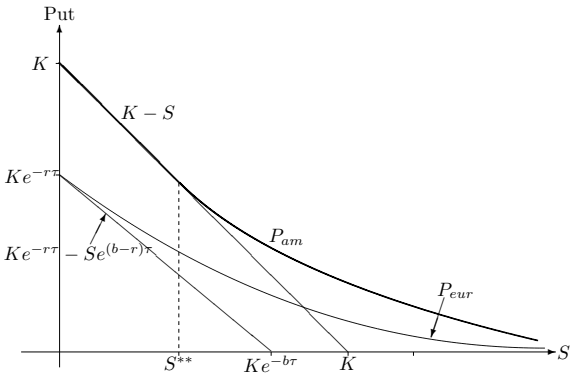



Figure 8.1: European put and early exercise of an American put with a continuous costs of carry $b$.

is higher than the possible increase of the option value due to the eventual underlying fluctuations. That is one of the reasons why the critical underlying price is dependent on time: $S^{**} = S^{**}(t)$.

From the derivation of the Black-Scholes differential equation it follows that it is valid as long as the option is not exercised. Given that there are no transaction costs in perfect markets, a revenue can be realized from an early exercise, which equals the intrinsic value of the option. One says in this case that the option falls back to its intrinsic value by early exercising. The Black-Scholes differential equation is valid where the underlying $S$ is either higher than the critical put-price $S^{**} = S^{**}(t)$ or lower than the critical call-price $S^{*} = S^{*}(t)$. The boundaries defined through $S^{**}(t)$ and $S^{*}(t)$ are largely unknown.

Figure 8.2 shows the regions where the option price $C = C(S,t)$ for an American call satisfies the Black-Scholes differential equation.

- In the interior $\{(S,t)|0 \leq S < S^{*}(t), t < T\}$ the Black-Scholes differential equation holds.
- At the boundaries $\{(S,t)|S = S^{*}(t), t < T\}$ and $\{(S,t)|0 \leq S, t = T\}$ the call falls back to the intrinsic value $\max(S-K,0)$.
- $C(S,t)$ and $\dfrac{\partial C(S,t)}{\partial S}$ are continuous in the whole region including the boundaries.

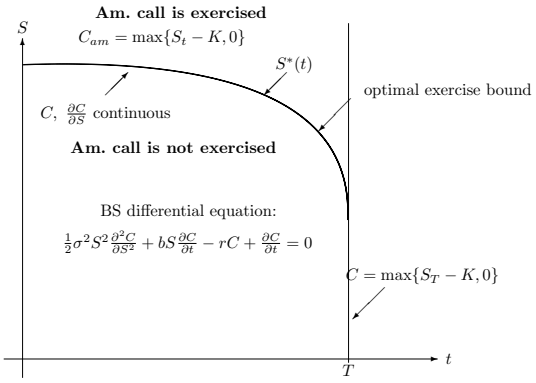



Figure 8.2: The exercise boundary $S^*(t)$ for an American call.

The numerical solution for such boundary problems is described in the next section. Based on the assumptions of perfect markets and the arbitrage free argument in Section 2.1 we derive some properties of American options without considering any specific mathematical models for the price process $S_t$.

**Theorem 8.1**

1. *An American call on an asset that does not yield any positive income before maturity is not early exercised and has the same value as a corresponding European call.*
2. *For an American call on an asset that generates positive income at discrete time points $t_1, \cdots, t_n$ the optimal exercise time can only lie just before one of these points. Consequently in the case of continuous positive payments, any time point can be an optimal exercise time.*

**Proof:**
Let $K$ denote the exercise price, $T$ the expiry date, $\tau = T - t$ the time to maturity of a call and $S_t$ the price of the underlying asset. $C_{am}(S, \tau)$ and $C_{eur}(S, \tau)$ denote the value of the respective American and European calls at time $t$ with time to maturity $\tau = T - t$ and current spot price $S_t = S$. Using the put-call parity for European options we get:

1. In the case of discrete dividends $D$ with the discounted value $D_t$ at time $t$, there should be costs, i.e. $D_t \leq 0$, by the no gain assumption.

Therefore, we get from Theorem 2.3:

$$C(S_t,\tau) = P(S_t,\tau) + S_t - D_t - Ke^{-r\tau} \geq S_t - Ke^{-r\tau} > S_t - K \quad (8.1)$$

where $C = C_{eur}$ and $P$ is the respective put price.
In the case of continuous dividends with rate $d \leq 0$, it follows from $b - r = -d \geq 0$ that:

$$\begin{aligned} C(S_t,\tau) &= P(S_t,\tau) + S_t e^{(b-r)\tau} - Ke^{-r\tau} \\ &\geq S_t e^{(b-r)\tau} - Ke^{-r\tau} > S_t - K \end{aligned} \quad (8.2)$$

In both cases we verify that $C(S_t,\tau) > S_t - K$ for European calls. Since $C_{am} \geq C$, we conclude that:

$$C_{am}(S_t,\tau) > S_t - K,$$

i.e. the value of an American call is always higher than the intrinsic value until maturity. Therefore early exercise is avoided.

2. Without any restriction we consider the case where $t_1$ is the next payment time. $\tilde{t} < t_1$ represents any time earlier. $\tilde{C}(S_t,\tilde{\tau})$ with $\tilde{\tau} = \tilde{t} - t$ is the value of a European call with the same exercise price $K$ but with a different maturity at $\tilde{t}$. Since there are no payments before $\tilde{t}$ at all, it follows from part 1 that $\tilde{C}(S_t,\tilde{\tau}) > S_t - K$ for $t < \tilde{t}$. Due to the longer time to maturity and the possibility of early exercise of American calls, it follows that:

$$C_{am}(S_t,\tau) \geq \tilde{C}(S_t,\tilde{\tau}) > S_t - K \quad (8.3)$$

As in part 1, the value of an American call at any time $t < \tilde{t}$ lies strictly above the intrinsic value, which excludes an early exercise. Since $\tilde{t} < t_1$ can take any value, $C_{am}(S_t,\tau)$ can fall to the intrinsic value only at time $t_1$ (or at a respectively later time point).

□

Figure 8.3 shows a graphical representation of the first part of the theorem.

- If $b \geq r$ which is equivalent to $d \leq 0$, then $C_{am} = C_{eur}$.
- If $b < r$ which is equivalent to $d > 0$, then $C_{am} > C_{eur}$.

It is also possible to derive a formula similar to the put call parity for American options. Given that, without a specific model, the critical prices $S^*(t)$, $S^{**}(t)$ and consequently the time point for early exercise are unknown, the formula is just an inequality.

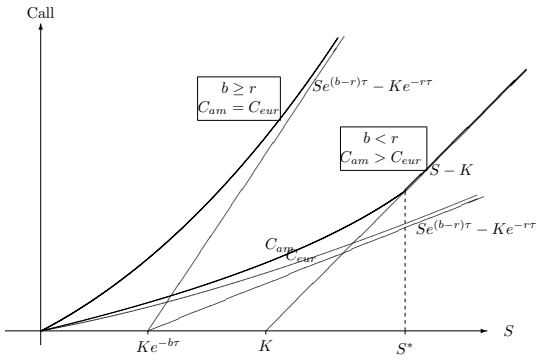



Figure 8.3: A European call and an early exercised American Call.

**Theorem 8.2 (Put-Call Parity for American options)**
*We consider an American call and an American put with the same maturity date $T$ and the same exercise price $K$ on the same underlying asset. Let $C_{am}(S,\tau)$ and $P_{am}(S,\tau)$ denote the option prices at time $t$ when the spot price is $S_t = S$ and the time to maturity is $\tau = T - t$. It holds*

1. *if there are incomes or costs during the time to maturity $\tau = T - t$ with the discounted value $D_t$ at time $t$, then*

$$P_{am}(S_t,\tau)+S_t-Ke^{-r\tau} \geq C_{am}(S_t,\tau) \geq P_{am}(S_t,\tau)+S_t-D_t-K \quad (8.4)$$

2. *if there are continuous costs of carry with rate $b$ on the underlying asset, then*

$$\begin{aligned} P_{am}(S_t,\tau) + S_t - Ke^{-r\tau} &\geq C_{am}(S_t,\tau) \\ &\geq P_{am}(S_t,\tau) + S_t e^{(b-r)\tau} - K \\ &\qquad \text{if } b < r \qquad (8.5) \\ P_{am}(S_t,\tau) + S_t e^{(b-r)\tau} - Ke^{-r\tau} &\geq C_{am}(S_t,\tau) \\ &\geq P_{am}(S_t) + S_t - K \text{ if } b \geq r \end{aligned}$$

**Proof:**
Supposing that, without any restriction, the underlying asset is a stock paying dividends $D_1$ at time $t_1$.

1. We show firstly, the left inequality. We consider a portfolio consisting of the following four positions:

| Position | $\tilde{t} = t_1 - \Delta t$ early exercise of the call | $t_1$ | $T$ Call is exercised early $S_T \le K$ | $S_T > K$ | $T$ Call is not exercised early $S_T \le K$ | $S_T > K$ |
|---|---|---|---|---|---|---|
| 1. | $\ge 0$ | – | $K - S_T$ | 0 | $K - S_T$ | 0 |
| 2. | $S_{\tilde{t}}$ | $D_1$ | – | – | $S_T$ | $S_T$ |
| 3. | $-Ke^{-r(T-\tilde{t})}$ | – | $-K$ | $-K$ | $-K$ | $-K$ |
| 4. | $-(S_{\tilde{t}} - K)$ | – | $Ke^{r(T-\tilde{t})}$ | $Ke^{r(T-\tilde{t})}$ | 0 | $-(S_T - K)$ |
| Sum | $\ge 0$ | $\ge 0$ | $\ge 0$ | $\ge 0$ | 0 | 0 |

Table 8.1: Portfolio value at time $\tilde{t}$, $t_1$ and $T$.

SFEamerican

1. buy an American put
2. buy a stock
3. sell bonds (i.e. borrow money) with the nominal value $K$ and the maturity date $T$
4. sell an American call

In this portfolio, position 1 is held until time $T$ even if this should be suboptimal, i.e. the put is not exercised early, even when the call holder exercises it early. Note from Theorem 8.1 that an early exercise of the call is only possible at time $\tilde{t} = t_1 - \Delta t$ where $\Delta t \approx 0$, i.e. directly before the payment at time $t_1$. In this case we deliver the stock of the portfolio. The value of the portfolio at time $T$ is given in the Table 8.1.

Therefore it holds for every time $t$ as claimed:

$$P_{am}(S_t, \tau) + S_t - Ke^{-r\tau} - C_{am}(S_t, \tau) \ge 0 \tag{8.6}$$

The proof of the second inequality is analogous but with opposite positions. Here we allow that the put can be exercised early, see Table 8.2.

1. buy an American call
2. sell a stock
3. buy a bond (i.e. lend money) at present value $K + D_t$ with maturity date $T$
4. sell an American put

Therefore we have $t$ for every time as claimed:

$$C_{am}(S_t, \tau) - P_{am}(S_t, \tau) - S_t + K + D_t \ge 0 \tag{8.7}$$

| Pos. | $\tilde{t} = t_1 - \Delta t$ early exercise of a put | $T$ Put is exercised early $S_T \leq K$ | Put is exercised early $S_T > K$ | Put is not exercised early $S_T \leq K$ | Put is not exercised early $S_T > K$ |
|---|---|---|---|---|---|
| 1. | $\geq 0$ | 0 | $S_T - K$ | 0 | $S_T - K$ |
| 2. | $-S_{\tilde{t}} - D_t e^{r(\tilde{t}-t)}$ | $-S_T$ $-D_t e^{r\tau}$ | $-S_T$ $-D_t e^{r\tau}$ | $-S_T$ $-D_t e^{r\tau}$ | $-S_T$ $-D_t e^{r\tau}$ |
| 3. | $(D_t + K)e^{r(\tilde{t}-t)}$ | $(D_t + K)e^{r\tau}$ | $(D_t + K)e^{r\tau}$ | $(D_t + K)e^{r\tau}$ | $(D_t + K)e^{r\tau}$ |
| 4. | $-(K - S_{\tilde{t}})$ | 0 | 0 | $-(K - S_T)$ | 0 |
| Sum | $\geq 0$ | $\geq 0$ | $\geq 0$ | $\geq 0$ | $\geq 0$ |

Table 8.2: Portfolio value at time $\tilde{t}$, $t_1$ and $T$.

2. For continuous cost of carry we first consider the case where $b \geq r \iff d \leq 0$. We prove the left inequality first:

$$P_{am}(S_t, \tau) + S_t e^{(b-r)\tau} - Ke^{-r\tau} \geq C_{am}(S_t, \tau) \tag{8.8}$$

Consider the following portfolio at time $t$:

1. buy an American put
2. buy $e^{(b-r)\tau}$ stocks
3. sell bonds at nominal value $K$ with date of expiring $T$
4. sell an American call

As in part 1 it follows that the value of the portfolio at time $T$ is zero if the call is *not* exercised early. The continuous costs of carry ($d \leq 0$) are financed through selling part of the stocks so that exactly one stock is left in the portfolio at time $T$.

If, on the other hand, the call is exercised early at time $\tilde{t}$, the whole portfolio is then liquidated and we get:

$$\begin{array}{lll} P_{am}(S_{\tilde{t}}, \tau) - (S_{\tilde{t}} - K) + S_{\tilde{t}} e^{(b-r)(T-\tilde{t})} - Ke^{-r(T-\tilde{t})} & = & \\ P_{am}(S_{\tilde{t}}, \tau) + K(1 - e^{-r(T-\tilde{t})}) + S_{\tilde{t}}(e^{(b-r)(T-\tilde{t})} - 1) & \geq & 0 \end{array} \tag{8.9}$$

The value of the portfolio at time $t$ is:

$$P_{am}(S_t, \tau) + S_t e^{(b-r)\tau} - Ke^{-r\tau} - C_{am}(S_t, \tau) \geq 0 \tag{8.10}$$

If $b < r \iff d > 0$ the left inequality is similarly proved,

$$P_{am}(S_t, \tau) + S_t - Ke^{-r\tau} \geq C_{am}(S_t, \tau) \tag{8.11}$$

where it is enough to hold one stock in the portfolio as $d > 0$.

We now show the right inequality for the case $b \geq r$

$$C_{am}(S_t, \tau) \geq P_{am}(S_t, \tau) + S_t - K \tag{8.12}$$

We consider the following portfolio at time $t$:

1. buy an American call
2. sell an American put
3. sell a stock (short sales)
4. buy a bond with nominal value $Ke^{r\tau}$ and expiring at time $T$

If the put is not exercised early, it holds at time $T$:

$$\begin{array}{ccccccccc} 0 & - & (K - S_T) & -S_T e^{-(b-r)\tau} + Ke^{r\tau} & \geq & 0 & \text{if} & S_T < K \\ (S_T - K) & + & 0 & -S_T e^{-(b-r)\tau} + Ke^{r\tau} & \geq & 0 & \text{if} & S_T \geq K \end{array} \tag{8.13}$$

If the put is exercised early at time $\tilde{t}$, the whole portfolio is liquidated and we get:

$$C_{am}(S_{\tilde{t}}, \tau) - (K - S_{\tilde{t}}) - S_{\tilde{t}} e^{-(b-r)(\tilde{t}-t)} + Ke^{r(\tilde{t}-t)} \geq 0 \tag{8.14}$$

Thus the value of the portfolio at time $t$ is:

$$C_{am}(S_t, \tau) - P_{am}(S_t, \tau) - S_t + K \geq 0 \tag{8.15}$$

Analogously one gets for the right inequality when $b < r$

$$C_{am}(S_t, \tau) \geq P_{am}(S_t, \tau) + S_t e^{(b-r)\tau} - K \tag{8.16}$$

where the position of the stock is reduced to $e^{(b-r)\tau}$ stocks. □

## 8.2 The Trinomial Model for American Options

The American option price can only be determined numerically. Similar to the European options, the binomial model after Cox-Ross-Rubinstein can be used. In this section we introduce a slightly more complex but numerically efficient approach based on trinomial trees, see Dewynne et al. (1993). It is related to the classical numerical procedures for solving partial differential equations, which are also used to solve the Black-Scholes differential equations.

The trinomial model (see Section 4.2) follows the procedure of the binomial model whereby the price at each time point $t_j = j\Delta t, j = 0, ..., n$, can change

to three, instead of two directions, with $\Delta t = T/n$, see Figure 8.4. The value $S_j^k$ at time $t_j$ can attain the values $u_1 \cdot S_j^k, u_2 \cdot S_j^k, u_3 \cdot S_j^k$ at $t_{j+1}$, where $u_i > 0, i = 1, 2, 3$, are suitably chosen parameters of the model. The probability with which the price moves from $S_j^k$ to $u_i \cdot S_j^k$ is represented as $p_i, i = 1, 2, 3$. The price process $S_j, j = 0, ..., n$, in discrete time is also a trinomial process, i.e. the logarithm of the price $Z_j = \log S_j, j = 0, ..., n$, is an ordinary trinomial process with possible increments $\log u_i, i = 1, 2, 3$.

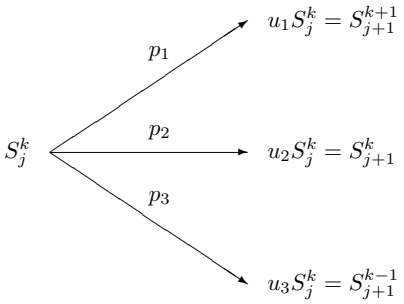



Figure 8.4: Possible price movements in the trinomial model.

As in the binomial model three conditions must be fulfilled: The sum of the probabilities $p_i, i = 1, 2, 3$, is one, the expectation and variance of the logarithmic increments $Z_j$ must be the same as those of the logarithms of the geometric Brownian motion over the time interval $\Delta t$. From these conditions we get three equations:

$$p_1 + p_2 + p_3 = 1, \tag{8.17}$$

$$p_1 \log u_1 + p_2 \log u_2 + p_3 \log u_3 = (b - \frac{1}{2}\sigma^2)\Delta t, \tag{8.18}$$

$$p_1(\log u_1)^2 + p_2(\log u_2)^2 + p_3(\log u_3)^2 = \sigma^2 \Delta t + (b - \frac{1}{2}\sigma^2)^2 \Delta t^2 \tag{8.19}$$

In the last equation we use $\mathsf{E}(Z_j^2) = Var(Z_j) + (\mathsf{E}\, Z_j)^2$. Since there are 6 unknown parameters in the trinomial model, we have the freedom to introduce three extra conditions in order to identify a unique and possibly simple solution of the system of equations. For a symmetric price tree, we require the recombination property

$$u_1 u_3 = u_2^2.$$

From this, the number of possible prices at time $t_n$ is reduced from maximally $3^n$ to $2n+1$ and consequently memory space and computation time are saved.

To determine the parameters of the model we still need two more conditions. We discuss two approaches of which one is motivated by binomial models while the other by numeric analysis of partial differential equations.

**a.)** The first approach requires that a time step of $\Delta t$ in the trinomial model corresponds to two time steps in the binomial model: $u_1$ represents two upwards increments, $u_3$ two downwards increments and $u_2$ one upward and one downward increment (or reversed). The binomial model fulfills the recombination condition $d = 1/u$. Since now that the length of the time step is $\Delta t/2$, it holds following Section 7.1

$$u = e^{\sigma\sqrt{\Delta t/2}}$$

and the probability for the price moving upwards in the binomial model is:

$$p = \frac{1}{2} + \frac{1}{2}(b - \frac{1}{2}\sigma^2)\frac{\sqrt{\Delta t/2}}{\sigma}.$$

Then we get as conditions for the parameters of the trinomial model

$$\begin{aligned} u_1 &= u^2,\ u_2 = 1,\ u_3 = u^{-2}, \\ p_1 &= p^2,\ p_2 = 2p(1-p),\ p_3 = (1-p)^2. \end{aligned}$$

With these parameters, the trinomial model performs as well as the corresponding binomial model for the European option, requiring however only half of the time steps. It converges therefore faster than the binomial model towards the Black-Scholes solution.

**Example 8.1** *Given the parameters from Table 7.1, the trinomial model provides a price of 30.769 for a European call option after $n = 50$ steps. This corresponds exactly to the value the binomial model attains after $n = 100$ steps, see Table 7.2.*

SFEBiTree

American options differ from the European options in that the options can be exercised at any time $t^*$ , $0 < t^* \leq T$. Consequently the value of a call falls back to the intrinsic value if it is early exercised:

$$C(S, t^*) = \max\{0, S(t^*) - K\}.$$

Mathematically, we have to solve a free boundary value problem which can only be done numerically.

$V_j^k$ denotes the option value at time $t_j$ if the spot price of stocks is $S_j = S_j^k$. As in the binomial model for European options we use $V_j^k$ to denote the

| | |
|---|---|
| spot stock price $S_t$ | 230.00 |
| exercise price $K$ | 210.00 |
| time to maturity $\tau$ | 0.50 |
| volatility $\sigma$ | 0.25 |
| interest rate $r$ | 0.04545 |
| dividend | no |
| steps | 50 |
| option type | American call |
| option price | 30.769 |

Table 8.3: The value of an American call option.

| | Call | Put |
|---|---|---|
| European | 30.769 | 6.051 |
| American | 30.769 | 6.212 |

Table 8.4: The value of an American call and put option.

discounted expectation that is calculated from the prices attainable in the next time step, $V_{j+1}^{k+1}$, $V_{j+1}^{k}$ and $V_{j+1}^{k-1}$. Different from the European options, the expectation of American options may not fall under the intrinsic value. The recursion for the American call price is thus:

$$C_j^k = \max\{S_j^k - K\,,\, e^{-r\Delta t}[p_1 C_{j+1}^{k+1} + p_2 C_{j+1}^{k} + p_3 C_{j+1}^{k-1}]\}.$$

**Example 8.2** *Table 8.3 gives the parameters and the value of an American call option determined with steps $n = 50$ in a trinomial model. Compatible with Theorem 8.1 it gives the same value 30.769 as a European option because the underlying stock yields no dividend before maturity.*

*The American put is on the other hand more valuable than the European. With the parameters from Table 8.3 one gets $P_{eur} = 6.05140$ and $P_{am} = 6.21159$. These results are presented in Table 8.4.*

**b.)** In the second approach the trinomial parameters $p_i$ and $u_i$ are determined through additional conditions. Here a certain upwards trend is shown in the whole price tree since we replace the condition $u_2 = 1$ by

$$u_2 = u \stackrel{\text{def}}{=} e^{(b-\frac{1}{2}\sigma^2)\Delta t}.$$

Furthermore we assume $p_1 = p_3$, therefore, the four above-mentioned conditions arise:

$$p_1 = p_3 = p,\ p_2 = 1 - 2p, \quad \text{with } p = \frac{\Delta t}{Th^2},$$

$$u_1 = ue^{\sigma h\sqrt{T/2}},\ u_2 = u,\ u_3 = ue^{-\sigma h\sqrt{T/2}},$$

where $h$ is another yet free parameter. The $p_i$ and $u_i$ fulfill the equation system (8.17) - (8.19) exactly. To guarantee that $p_1, p_2, p_3$ are not trivial probabilities, i.e they must be between 0 and 1 and $0 < p < 1/2$, $h$ must fulfill the following condition:

$$h > \sqrt{\frac{2\Delta t}{T}}. \tag{8.20}$$

We consider now a European option. Here the trinomial model delivers the following recursion for the possible option values depending on the probabilities $p_i$ and the change rates $u_i$:

$$V_j^k = e^{-r\Delta t}\left(\frac{\Delta t}{Th^2}V_{j+1}^{k+1} + (1 - 2\frac{\Delta t}{Th^2})V_{j+1}^k + \frac{\Delta t}{Th^2}V_{j+1}^{k-1}\right). \tag{8.21}$$

We consider $\Delta t = -(T - t_{j+1}) + (T - t_j)$ for all $j = 0, ..., n-1$, and we put $h^* = \Delta t/T$ as well as

$$Z_j^k = V_j^k e^{-r(T-t_j)},\ Z_{j+1}^k = V_{j+1}^k e^{-r(T-t_{j+1})}.$$

The recursion (8.21) for the option values $V_j^k$ then becomes

$$\frac{Z_j^k - Z_{j+1}^k}{h^*} = \frac{Z_{j+1}^{k+1} - 2Z_{j+1}^k + Z_{j+1}^{k-1}}{h^2}. \tag{8.22}$$

This is the explicit difference approximation of the parabolic differential equation (6.15), see Samaskij (1984). The condition (8.20) corresponds to the well-known stability requirement for explicit difference schemes. Compared to the previously discussed approach, the probabilities $p_i$ in this variant of the trinomial model and the calculation in (8.21) are not dependent on the volatility. The recursion (8.21) depends on $\sigma$ only through the initial condition $S_n = S_T$, i.e. on the price of a stock following a geometric Brownian motion with volatility $\sigma$.

## 8.3 Recommended Literature

American options are classic financial derivative instruments and play a central role in the literatures referenced in Chapter 6. The trinomial model as numerical procedures to approximate the option prices is introduced in detail in Dewynne et al. (1993) and Wilmott et al. (1995).

# 9 Exotic Options

A whole series of complex, so-called exotic options exist, and are mainly used in OTC-trading (over the counter) to meet the special needs of corporate customers. The most important types of Exotic Options are:

- Compound Options
- Chooser Options
- Barrier Options
- Asian Options
- Lookback Options
- Cliquet Options

Figure 9.1 shows an example for DAX warrants issued by Sal.Oppenheim. These products payout at maturity a fraction of the value of a barrier option. The value of a barrier option is given by the value of a corresponding European option if the barrier was not crossed. As there are European calls and puts there are two types of warrants: Long and Short. In the example, the strike of the European options and the barrier (StopLoss) coincide. Moreover, the warrants all payoff 1% of the value of the barrier options as the last but one column shows. The maturity is given in the last column of the table.

## 9.1 Compound Options, Option on Option

With a compound option one has the right to acquire an ordinary option at a later date. To illustrate such a compound option consider a *Call-on-a-Call* with the parameters:

$$\begin{array}{c} \text{maturity dates } T_1 < T_2 \\ \text{strike prices } K_1, K_2. \end{array}$$

Such a compound option gives the owner the right at time $T_1$ for the price $K_1$ to buy a call that has a maturity date $T_2$ and a strike price $K_2$.

Basiswert: DAX 4.130,81 +41,68 +1,02% 11.11.2004 Java-Applet: aktiv Neu Starten

| WKN | Typ | Bid | Zeit | Ask | Zeit | Strike | StopLoss | Währung | BV | Fälligkeit |
|---|---|---|---|---|---|---|---|---|---|---|
| SAL60F | Long | 2.470 | 7:05:14 PM | 2.490 | 7:05:14 PM | 3.900,00 | 3.900,00 | XXP | 0,01 | 23.12.2004 |
| SAL6UC | Long | 2.650 | 7:05:24 PM | 2.670 | 7:05:24 PM | 3.900,00 | 3.900,00 | XXP | 0,01 | 24.03.2005 |
| SAL60G | Long | 2.240 | 7:05:14 PM | 2.260 | 7:05:14 PM | 3.925,00 | 3.925,00 | XXP | 0,01 | 23.12.2004 |
| SAL6U9 | Long | 1.970 | 7:05:14 PM | 1.990 | 7:05:14 PM | 3.950,00 | 3.950,00 | XXP | 0,01 | 23.12.2004 |
| SAL6UA | Long | 1.730 | 7:05:14 PM | 1.750 | 7:05:14 PM | 3.975,00 | 3.975,00 | XXP | 0,01 | 23.12.2004 |
| SAL6UB | Long | 1.470 | 7:05:14 PM | 1.490 | 7:05:14 PM | 4.000,00 | 4.000,00 | XXP | 0,01 | 23.12.2004 |
| SAL4VM | Short | 0.160 | 7:05:14 PM | 0.180 | 7:05:14 PM | 4.150,00 | 4.150,00 | XXP | 0,01 | 23.12.2004 |
| SAL4VN | Short | 0.410 | 7:05:14 PM | 0.430 | 7:05:14 PM | 4.175,00 | 4.175,00 | XXP | 0,01 | 23.12.2004 |
| SAL1S6 | Short | 0.660 | 7:05:24 PM | 0.670 | 7:05:24 PM | 4.200,00 | 4.200,00 | XXP | 0,01 | 23.12.2004 |
| SAL2GN | Short | 0.610 | 7:05:27 PM | 0.630 | 7:05:27 PM | 4.200,00 | 4.200,00 | XXP | 0,01 | 24.03.2005 |

Figure 9.1: Basic information on DAX warrants issued by Sal.Oppenheim.

The value $V(S,t)$ of this option at time $t$ with an actual price $S_t = S$ can be calculated by applying the Black-Scholes formula twice:

1) Beginning at time $t = T_2$, calculate (explicitly or numerically) the value of the call, which can be bought at time $T_1$, at time $t = T_1$. This value is $C(S, T_1)$.
2) The purchasing option of the call at time $T_1$ is only exercised when

$$C(S, T_1) > K_1$$

Thus it holds that

$$V(S, T_1) = \max\{C(S, T_1) - K_1,\ 0\}.$$

Calculate $V(S,t)$ for $t < T_1$ using the Black-Scholes equation with these restrictions at $t = T_1$ analog to the normal call.

**Remark 9.1** *Since $V(S,t)$ is only dependent on $t$ and the price $S_t$ at time $t$, this value function fulfills the Black-Scholes equation so that our approach is justified.*

We illustrate the pricing of a compound option in the binomial tree model of Cox, Ross and Rubinstein. We consider an asset with an initial value of 230 and a volatility of 25%. The corresponding tree for one year with time steps $\Delta t = 0.2$ is shown in table 9.1. The corresponding prices $C$ of a European call with a strike of $K_2 = 210$ that expires in $T_2 = 1$ year are presented in table 9.1. The value of a compound option on this call with compound expiration at $T_1 = 0.4$ and compound strike $K_1 = 20$ is shown in table 9.1.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | 402.26 |
| | | | | | 359.71 | | |
| | | | | 321.66 | | | 321.66 |
| | | | 287.63 | | 287.63 | | |
| | | 257.21 | | 257.21 | | | 257.21 |
| 230 | | | 230 | | 230 | | |
| | | 205.67 | | 205.67 | | | 205.67 |
| | | | 183.91 | | 183.91 | | |
| | | | | 164.46 | | | 164.46 |
| | | | | | 147.06 | | |
| | | | | | | | 131.51 |

Table 9.1: Binomial tree of a stock

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | 192.25 |
| | | | | | 151.60 | | |
| | | | | 115.42 | | | 111.66 |
| | | | 83.75 | | 79.53 | | |
| | | 58.14 | | 51.99 | | | 47.21 |
| 38.91 | | | 32.30 | | 23.98 | | |
| | | 19.39 | | 12.18 | | | |
| | | | 6.19 | | | | |

Table 9.2: Binomial tree of a call option

| | | |
|---|---|---|
| | | 63.75 |
| | 38.32 | $\max\{C^{uu} - K_1; 0\}$ |
| 22.49 | PV(63.75, 12.30) | 12.30 |
| PV(38.32,6.25) | 38.32 | $\max\{C^{ud} - K_1; 0\}$ |
| | PV(12.30, 0) | 0 |
| | | $\max\{C^{dd} - K_1; 0\}$ |

Table 9.3: Binomial tree of a call on a call option

## 9.2 Chooser Options or "As You Wish" Options

A *Chooser Option* is a form of the compound option, where the buyer can decide at a later date which type of option he would like to have. To illustrate this, consider a *regular Chooser Option* with the parameters:

$$\begin{array}{c}\text{maturity dates } T_1 < T_2\\ \text{strike prices } K_1, K_2.\end{array}$$

This option gives the right at time $T_1$ for the price $K_1$ to buy a call or a put (as preferred), which has a maturity $T_2$ and a strike price $K_2$: in the language of compound options this is referred to as a *Call-on-a-Call* or *Put*.

The value $V(S,t)$ can be found by applying the Black-Scholes formula three times:

1) Determine the value $C(S,T_1)$ and $P(S,T_1)$ of a call and a put with a maturity $T_2$, and strike price $K_2$.
2) Solve the Black-Scholes equation for $t < T_1$ with the restriction

$$V(S,T_1) = \max\{C(S,T_1) - K_1,\ P(S,T_1) - K_1, 0\}$$

## 9.3 Barrier Options

A *Barrier Option* changes its value in leaps as soon as the stock price reaches a given barrier, which can also be time dependent. As an example consider a simple European barrier option which at

$$\begin{array}{c}\text{maturity } T\text{, strike price } K \text{ and}\\ \text{barrier } B\end{array}$$

gives the holder the right to buy a stock at time $T$ for the price $K$ provided that

- *down-and-out:* $S_t > B$ for all $0 \le t \le T$
- *up-and-out:* $S_t < B$ for all $0 \le t \le T$

This type of *Knock-out Option* is worthless as soon as the price $S_t$ reaches the barrier. Figure 9.2 represents the situation for a down-and-out option

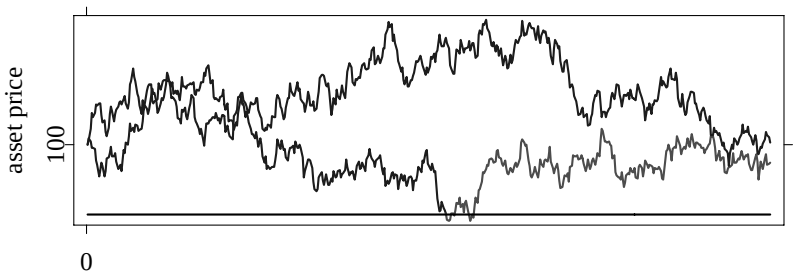



Figure 9.2: Two possible paths of the asset price. When the price hits the barrier (lower path), the option expires worthless.

with two possible paths of the asset price. When the price hits the barrier (lower path), the option expires worthless regardless of any further evolution of the price.

A *Knock-in Option* is just the opposite to the knock-out option. It is worthless up until the barrier is reached.

For example, a European *Knock-in Call* consists of the right to buy stock provided that

- *down-and-in:* $S_t \leq B$ for some $0 \leq t \leq T$
- *up-and-in:* $S_t \geq B$ for some $0 \leq t \leq T$

The value of a barrier option is no longer dependent on a stock price at a specific point in time, but on the overall development of the stock price during the option's life span. Thus in principle it does not fulfill the Black-Scholes differential equation. The dependence however, is essentially simple enough to work with the conventional Black-Scholes application. As an example consider a *Down-and-out Call* with $K > B$. As long as $S_t > B$, $V(S,t)$ fulfills the Black-Scholes equation with the restriction:

$$V(S,T) = \max(S_T - K,\ 0)$$

In the event that the price reaches the barrier $B$, the option of course becomes worthless:

$$V(B,t) = 0\ , \qquad 0 \leq t \leq T,$$

is therefore an additional restriction that needs to be taken into account when solving the Black-Scholes differential equation. The explicit solution is given as:

$$V(S,t) = C(S,t) - \left(\frac{B}{S}\right)^{\alpha} C\left(\frac{B^2}{S}, t\right)$$

with $\alpha = \frac{2r}{\sigma^2} - 1$, where $C(S,t)$ represents the value of a common European call on the stock in question. The value $\bar{V}(S,t)$ of a European *Down-and-in Call* can be calculated analogously. If one already knows $V(S,t)$, one can also use the equation

$$\bar{V}(S,t) + V(S,t) = C(S,t).$$

It is fulfilled since a down-and-in and a down-and-out call together have the same effect as a normal call.

## 9.4 Asian Options

With Asian options the value depends on the average stock price calculated over the entire life span of the option. With an *Average Strike Option* over the time period $0 \leq t \leq T$ the payoff, for example, has at maturity the form

$$\max\left(S_t - \frac{1}{t}\int_0^t S_s\, ds\,,\ 0\right), t = T.$$

With an *American Average Strike Option* this is also the payoff when the option is exercised ahead of time at some arbitrary time $t \leq T$.

To calculate the value of an *Asian Option* consider a general class of European options with a payoff at time $T$ that is dependent on $S_T$ and $I_T$ with

$$I_t = \int_0^t f(S_s, s)ds.$$

Analogous to the Black-Scholes equation we derive an equation for the value at time $t$ of such a *path dependent option* $V(S,I,t)$. At time $t$ with a stock price $S_t$ this results in

$$\begin{aligned} I_t + dI_t &\overset{\text{def}}{=} I_{t+dt} = \int_0^{t+dt} f(S_s, s)ds \\ &= I_t + \int_t^{t+dt} f(S_s, s)dt \\ &= I_t + f(S_t, t)dt + \mathcal{O}(dt). \end{aligned}$$

Thus the differential of $I_t$ is equal to $dI_t = f(S_t, t)dt$.

Using Itôs Lemma it follows for $V_t = V(S_t, I_t, t)$ that:

$$dV_t = \sigma S_t \frac{\partial V_t}{\partial S} dW_t + f(S_t, t)\frac{\partial V_t}{\partial I} dt$$

$$+\left(\frac{1}{2}\sigma^2 S^2\,\frac{\partial^2 V_t}{\partial S}+\frac{\partial V_t}{\partial t}\right)dt$$

Analogous to the derivation of the Black-Scholes formula continuous delta hedging produces a risk free portfolio from an option and $\Delta_t = \partial V_t/\partial S$ sold stocks. Together with the restriction of no arbitrage it follows for the case of no dividends ($b = r$) that:

$$\frac{\partial V_t}{\partial t}+f(S_t,t)\frac{\partial V_t}{\partial I}+\frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t}{\partial S^2}+rS_t\frac{\partial V_t}{\partial S}-r\ V_t=0$$

This is the Black-Scholes equation with an additional term $f(S_t,t)\frac{\partial V_t}{\partial I}$. The boundary condition in this case is

$$V(S,I,T)=g(S,I,T).$$

For an *Average Strike Call* we have:

$$g(S,I,t)=\max(S-\frac{1}{t}I,\ 0)\quad\text{und}\quad f(S,t)=S.$$

For European options an explicit analytic solution of the differential equation exists in which complicated, specialized functions appear, the so called confluent hypergeometric functions. The numerical solution, however, is easier and faster to obtain.

The integral $\int_0^t S_s ds$ in practice is calculated as the sum over all quoted prices, for example, at 30 second time intervals. Discrete time Asian options use, in place of this, a substantially larger time scale. $I_t$ changes only once a day or once a week:

$$I_t=\sum_{j=1}^{n(t)} S_{t_j}\ ,\ \ t_{n(t)}\le t<t_{n(t)+1}$$

with $t_{j+1}-t_j = 1$ day or $= 1$ week and closing price $S_{t_j}$.

Such a discrete time Asian option corresponds largely to a common option with discrete dividend payments at time periods $t_1, t_2, \ldots$. From the assumption of no arbitrage follows a continuity restriction at $t_j$:

$$V(S,I,t_j-)=V(S,I+S,\ t_j+)$$

To determine the value of the option one begins as usual at the time of maturity where the value of the option is known:

1) $T = t_n$

$$V(S, I, T) = \max(S - \frac{1}{T} I_T,\ 0)$$

$$I_T = \sum_{j=1}^{n} S_{t_j}$$

Solve the Black-Scholes equation backwards to time $t_{n-1}$ and obtain

$$V(S, I + S,\ t_{n-1}+)$$

2) Calculate using the continuity restriction the new terminal value $V(S, I, t_{n-1}-)$. Solve the Black-Scholes equation backwards to time $t_{n-2}$ and obtain

$$V(S, I + S,\ t_{n-2}+)$$

etc.

## 9.5 Lookback Options

The value of a lookback option depends on the maximum or minimum of the stock price over the entire life span of the option, for example, a *Lookback Put* over the time period $0 \le t \le T$ has at maturity the payoff

$$\max(M_T - S_T,\ 0) \text{ with} \qquad M_t = \max_{0 \le s \le t} S_s.$$

To calculate the value of such an option first consider a path dependent option with

$$I_t(n) = \int_0^t S_s^n ds\,, \quad \text{i.e.} \quad f(S,t) = S^n.$$

In the sample path of Figure 9.3, we see a simulated price process $(S_t)$ as a solid line and the maximum process $(M_t)$ as increasing dotted line. The value of a lookback put is then given by the difference of the end value of the increasing curve and the final value of the price process.

With $M_t(n) = \{I_t(n)\}^{\frac{1}{n}}$ it holds that:

$$M_t = \lim_{n \to \infty} M_t(n).$$

From the differential equation for $I_t(n)$ and $n \to \infty$ it follows that the value $V_t = V(S_t, M_t, t)$ of a European lookback put fulfills the following equation:

$$\frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S^2} + r\ S_t \frac{\partial V_t}{\partial S} - r\ V_t = 0$$

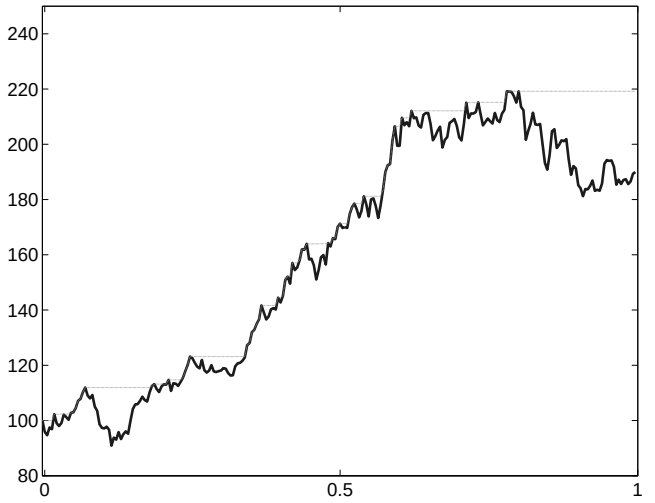



Figure 9.3: Sample paths of price process and corresponding maximum process.

This is the normal Black-Scholes equation. $M$ only appears as an argument of $V$ and in the boundary condition:

$$V(S, M, T) = \max(M - S,\ 0)$$

The solution is for a remaining time period of $\tau = T - t$, $\alpha = 2r/\sigma^2$:

$$\begin{aligned} V(S, M, t) &= S\left(\Phi(y_1) \cdot (1 + \frac{1}{\alpha}) - 1\right) \\ &\quad + M\ e^{-r\tau}\left(\Phi(y_3) - \frac{1}{\alpha}\left(\frac{M}{S}\right)^{\alpha-1}\Phi(y_2)\right) \end{aligned}$$

$$\text{with} \quad y_1 = \frac{1}{\sigma\sqrt{\tau}}\{\ln\frac{S}{M} + (r + \frac{1}{2}\sigma^2)\tau\}$$

$$y_2 = \frac{1}{\sigma\sqrt{\tau}}\{\ln\frac{S}{M} - (r + \frac{1}{2}\sigma^2)\tau\}$$

$$y_3 = \frac{1}{\sigma\sqrt{\tau}}\{\ln\frac{M}{S} - (r + \frac{1}{2}\sigma^2)\tau\}$$

## 9.6 Cliquet Options

A cliquet option settles periodically and resets the strike at the level of the underlying during the time of settlement. It is therefore a series of at-the-money options, but where the total premium is determined in advance. A cliquet option can be thought of as a series of "pre-purchased" at-the-money options. The payoff on each option can either be paid at the final maturity, or at the end of each reset period. The number of reset periods is determined by the buyer in advance. All other equal, more resets make the option more expensive.

The payoff of the cliquet call at maturity $T$ is given by

$$\max\{(S_{t_0}, S_{t_1}, ..., S_{t_n=T}) - S_{t_0}\}$$

and the payoff of the cliquet put is given by

$$\max\{S_{t_0} - (S_{t_0}, S_{t_1}, ..., S_{t_n=T})\}.$$

**Example 9.1**
*Consider a cliquet call with maturity $T = 3$ years and strike price $K_1 = 100$ in the first year. If in the first year the underlying were $S_1 = 90$, the cliquet option would expire worthless in the first year. The new strike price for the second year will be set to $K_2 = 90$. Assume, in the second year $S_2 = 120$, then the contract holder will receive a payoff and the strike price would reset to the new level of $K_3 = 120$ for the third year. Thus, higher volatility provides better conditions for investors to earn profits. Prices of cliquet options obey in a Black Scholes model a partial differential equation that has one variable, in addition to the standard variables of the Black-Scholes partial differential equation.*

## 9.7 Recommended Literature

Exotic options such as bond options are discussed in detail in Hull (2000). The numerical methods necessary for valuing complex path-dependent derivatives are briefly reviewed in Wilmott et al. (1995) and are discussed in more detail in Dewynne et al. (1993).

# 10 Models for the Interest Rate and Interest Rate Derivatives

As discussed in the last chapter, the important features of the Black and Scholes application are:

- constant risk free domestic interest rate $r$ (approximately fulfilled by stock options with life spans of $\leq 9$ months)
- independence of the price of the option's underlying from the interest rate $r$ (empirical research shows that for stocks this is approximately fulfilled).

Both features are violated for *bond options* and the longer time periods which are typically found in the analysis of these options.

A *bond* produces at the time of maturity $T$ a fixed amount $Z$, the nominal value, and if applicable at predetermined dates before $T$ dividend payments (*coupon*). If there are no coupons, the bond is referred to as a zero coupon bond or *zero bond* for short.

When valuing a bond option, coupons can be treated as discrete dividend payments when valuing stock options.

## 10.1 Bond Value with Known Time Dependent Interest Rate

To begin with we calculate the bond value $V(t)$ at time $t$ with a time dependent but known interest rate $r(t)$.

From the assumption of no arbitrage we conclude that a bond's change in value over the time period $[t, t+dt]$ with possible coupon payments $K(t)dt$ coincides with the change in value of a bank account with a value $V(t)$ and with an interest of $r(t)$ :

$$V(t+dt) - V(t) = \left(\frac{dV}{dt} + K(t)\right) dt = r(t)V(t)\ dt$$

Together with the boundary restrictions $V(T) = Z$ it follows that:

$$V(t) = \exp\{I(t)\}\{Z + \int_t^T K(s)\exp\{I(t)\}\,ds\} \tag{10.1}$$

with $I(t) = -\int_t^T r(s)ds$.

For a zero bond this simplifies to: $V(t) = Z \cdot \exp\{I(t)\}$

## 10.2 Stochastic Interest Rate Model

Due to the uncertainty associated with the future development of the interest rate, $r(t)$ is modelled as a random variable. In order to have an unambiguous, fixed interest rate, one usually considers the interest rate of an investment over the shortest possible time period:

$r(t) =$ *spot rate* = interest rate for the shortest possible investment.

$r(t)$ does not follow a geometric Brownian motion so that the Black-Scholes approach cannot be used. This is shown in Figure 10.1 that presents the 1-week Euribor rates as approximations for the spot rate. There are a number of models for $r(t)$, that are special cases of the following general proposal which models the interest rate as a general Itô Process:

$$dr(t) = \mu\{r(t),t\}dt + \sigma\{r(t),t\}dW_t \tag{10.2}$$

$\{W_t\}$ represents as usual a standard Wiener process.

Three most often used models are simple, special cases, where the coefficient functions of the models are

- *Vasicek*:

$$\mu(r,t) = a(b-r)\ ,\ \ \sigma(r,t) = \sigma \qquad a,b,\sigma \text{ constants} \tag{10.3}$$

- *Cox, Ingersoll, Ross:*

$$\mu(r,t) = a(b-r),\ \sigma(r,t) = \sigma\sqrt{r} \qquad a,b,\sigma \text{ constants} \tag{10.4}$$

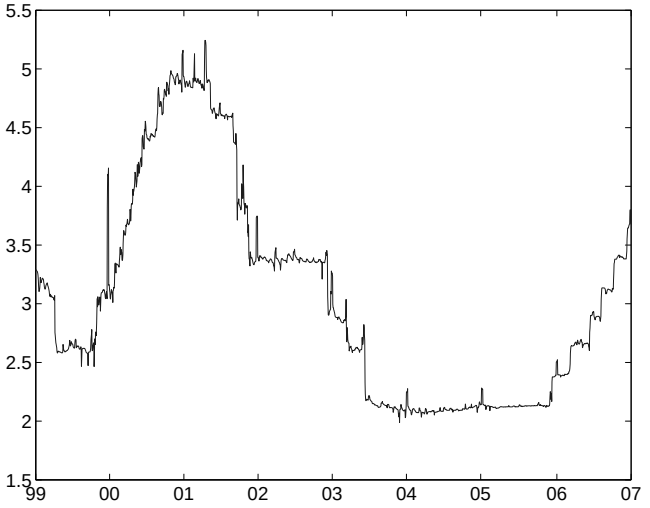



Figure 10.1: 1-week Euribor as approximation for the interest rate spot rate.

- *Hull, White:*

$$\mu(r,t) = \delta(t) - ar, \quad \sigma(r,t) = \sigma \qquad (10.5)$$
$$a, \sigma \text{ constants}, \ \delta(t) \text{ from market data}$$
$$\text{deducible known function}$$

In general $\mu(r,t)$ and $\sigma(r,t)$ can be conveniently chosen, in order to replicate empirically observed phenomena in the model. Below we will refer to $\sigma(r,t)$ as $w(r,t)$, in order to clearly differentiate between the function $w \stackrel{\text{def}}{=} \sigma$ and the constant $\sigma$, which appears as a parameter in the three models mentioned above.

## 10.3 The Bond Valuation Equation

A stock option can be hedged with stocks, and Black and Scholes use this in deriving the option pricing formula. Since there is no underlying financial instrument associated with a bond, bonds with varying life spans have to be used to mutually hedge each other, in order to derive the equation for valuation of bonds.

Consider a portfolio made up of a zero bond with a remaining life time of $\tau_1$ and $-\Delta$ zero bonds (i.e., $\Delta$ sold zero bonds) with a remaining life time of $\tau_2$. The value of the portfolio at time $t$ for the current interest rate $r(t) = r$ is:

$$\Pi(r,t) = V_1(r,t) - \Delta \cdot V_2(r,t).$$

where $V_i, i = 1,2$ stands for the value function of both bonds. We write $\Pi_t = \Pi\{r(t),t\}, V_{it} = V_i\{r(t),t\}, i = 1,2,\ \mu_t = \mu\{r(t),t\},\ w_t = w\{r(t),t\}$. Using Itôs Lemma it follows that:

$$d\Pi_t = \frac{\partial V_{1t}}{\partial t}dt + \frac{\partial V_{1t}}{\partial r}dr(t) + \frac{1}{2}w_t^2 \cdot \frac{\partial^2 V_{1t}}{\partial r^2}dt$$
$$-\Delta\left(\frac{\partial V_{2t}}{\partial t}dt + \frac{\partial V_{2t}}{\partial r}dr(t) + \frac{1}{2}w_t^2\frac{\partial^2 V_{2t}}{\partial r^2}dt\right)$$

By hedging the risks the random component disappears. This is achieved by choosing

$$\Delta = \frac{\partial V_{1t}}{\partial r}\Big/\frac{\partial V_{2t}}{\partial r}.$$

By insertion and by comparison of the portfolio with a risk free investment and taking advantage of the no arbitrage assumption, that is the equality of the change in value of portfolio and bond:

$$d\Pi_t = r(t) \cdot \Pi_t\, dt,$$

produces altogether

$$\left(\frac{\partial V_{1t}}{\partial t} + \frac{1}{2}w_t^2\frac{\partial^2 V_{1t}}{\partial r^2} - r(t)\ V_{1t}\right)\Big/\frac{\partial V_{1t}}{\partial r}$$
$$= \left(\frac{\partial V_{2t}}{\partial t} + \frac{1}{2}w_t^2\frac{\partial^2 V_{2t}}{\partial r^2} - r(t)\ V_{2t}\right)\Big/\frac{\partial V_{2t}}{\partial r}. \tag{10.6}$$

This is only correct when both sides are independent of the remaining life times $\tau_1, \tau_2$. $V_{1t}, V_{2t}$ therefore both satisfy the following differential equation

$$\frac{\partial V_t}{\partial t} + \frac{1}{2}w_t^2\frac{\partial^2 V_t}{\partial r^2} - r(t)\ V_t = -a\{r(t),t\} \cdot \frac{\partial V_t}{\partial r}$$

for some function $a(r,t)$ which is independent of the remaining time to maturity. With the economically interpretable variable

$$\lambda(r,t) = \frac{\mu(r,t) - a(r,t)}{w(r,t)}$$

this produces with the abbreviation $\lambda_t = \lambda\{r(t), t\}$ the *zero bond valuation equation* for $V(r, t)$ :

$$\frac{\partial V(r,t)}{\partial t} + \frac{1}{2} w_t^2 \frac{\partial^2 V(r,t)}{\partial r^2} - (\mu_t - \lambda_t w_t) \frac{\partial V(r,t)}{\partial r} - r(t)\ V(r,t) = 0 \quad (10.7)$$

with the boundary condition $V(r, T) = Z$ at the time of maturity and with additional boundary condition dependent on $\mu, w$. It should be noted that in the equation $\mu_t, w_t, \lambda_t$ stand for functions of $r$ and $t$.

The value $\lambda(r, t)$ has the following interpretation. Consider a risky portfolio, that is not hedged, consisting of a bond with the value $V_t = V\{r(t), t\}$. For the change in value within the time period $dt$ we obtain using Itôs Lemma and the zero bond valuation equation:

$$dV_t = r(t)\ V_t\ dt + w_t \cdot \frac{\partial V_t}{\partial r}(dW_t + \lambda\{r(t), t\}\, dt)$$

Since $\mathsf{E}[dW_t] = 0$, the mean change in value $\mathsf{E}[dV_t]$ is

$$\left(w_t \frac{\partial V_t}{\partial r}\right) \cdot \lambda\{r(t), t\}\, dt$$

above the increase in value $r(t)\ V_t\ dt$ of a risk free investment. $\lambda(r, t)\, dt$ is therefore the bonus on the increase in value, which one receives at time $t$ with a current interest rate $r(t) = r$ for taking on the associated risk. $\lambda(r, t)$ is thus interpreted as the *market price of risk*.

## 10.4 Solving the Zero Bond Valuation Equation

Consider the special case:

$$w(r, t) = \sqrt{\alpha(t) r - \beta(t)}$$

$$\mu(r, t) = -\gamma(t) \cdot r + \eta(t) + \lambda(r, t) w(r, t).$$

Inserting the solution assumption

$$V(r, t) = Z \cdot \exp\{A(t) - rB(t)\}$$

into the zero bond valuation equation results in the two equations

$$\frac{\partial A(t)}{\partial t} = \eta(t)\ B(t) + \frac{1}{2}\, \beta(t)\ B^2(t)$$

$$\frac{\partial B(t)}{\partial t} = \frac{1}{2}\alpha(t)\ B^2(t) + \gamma(t)\ B(t) - 1$$

with boundary condition $A(T) = B(T) = 0$ (since $V(r,T) = Z$).

For time independent $\alpha, \beta, \gamma, \eta$ there is an explicit solution, which with a remaining life time of $\tau = T - t$ has the form

$$B(t) = \frac{2(e^{\psi_1 \tau} - 1)}{(\gamma + \psi_1)(e^{\psi_1 \tau} - 1) + 2\psi_1}\,, \quad \psi_1 = \sqrt{\gamma^2 + \alpha}$$

$$\frac{2}{\alpha}A(t) = b_2\psi_2 \ln(b_2 - B) + (\psi_2 - \frac{\beta}{2})b_1 \ln(\frac{B}{b_1} + 1)$$

$$+\frac{1}{2}B\,\beta - b_2\psi_2 \ln b_2$$

with

$$b_{1/2} = \frac{\pm\gamma + \sqrt{\gamma^2 + 2\alpha}}{\alpha},\ \psi_2 = \frac{\eta + b_2\beta/2}{b_1 + b_2}$$

**Choice of parameters:**

1) The spot rate volatility is $\sqrt{\alpha r(t) - \beta}$. With this relation, $\alpha, \beta$ can be estimated from historical data, in a fashion similar to the historical volatility of stocks.
2) Taking the *yield curve* (see Section 11.4.3) into consideration, the discussion of which goes beyond the scope of this section, estimators for $\gamma$ and $\eta$ can be derived.

## 10.5 Valuation of Bond Options

As an example consider a European call with a strike price $K$ and a maturity $T_C$ on a zero bond with a maturity of $T_B > T_C$, i.e., the right is given to buy the bond at time $T_C$ at a price $K$.

$V_B(r,t) =$ value of the bond at time $t$ with the current interest rate $r(t) = r$

$C_B(r,t) =$ value of the call at time $t$ with the current interest rate $r(t) = r$

$C_B$ is only dependent on the random variable $r(t)$ and time $t$ and therefore itself also satisfies the zero bond's value equation, but with the boundary restrictions

$$C_B(r, T_C) = \max\left(V_B(r, T_C) - K,\ 0\right).$$

This equation, analogous to the corresponding Black-Scholes equation can be numerically solved.

## 10.6 Recommended Literature

The classical stochastic interest rate models are introduced in Vasicek (1977), Cox, Ingersoll and Ross (1985) and Hull and White (1990). A standard work taking a modern point of view of the interest rate in financial mathematics is Heath, Jarrow and Morton (1992).

# Part II

# Statistical Models of Financial Time Series

# 11 Introduction: Definitions and Concepts

Financial markets can be regarded from various points of view. First of all there are economic theories which make assertions about security pricing; different economic theories exist in different markets (currency, interest rates, stocks, derivatives, etc.). Well known examples include the purchasing power parity for exchange rates, interest rate term structure models, the *capital asset pricing model* (CAPM) and the Black-Scholes option pricing model. Most of these models are based on theoretical concepts which, for example, involve the formation of expectations, utility functions and risk preferences. Normally it is assumed that individuals in the economy act 'rationally', have rational expectations and are averse to risk. Under these circumstances prices and returns can be determined in equilibrium models (as, for example, the CAPM) which clear the markets, i.e., supply equals aggregate demand. A different Ansatz pursues the arbitrage theory (for example, the Black-Scholes model), which assumes that a riskless profit would be noticed immediately by market participants and be eliminated through adjustments in the price. Arbitrage theory and equilibrium theory are closely connected. The arbitrage theory can often get away with fewer assumptions, whereas the equilibrium theories reach more explicitly defined solutions for complex situations.

The classical econometric models are formulated with the economically interpreted parameters. One is interested in the following empirical questions:

1. How well can a specific model describe a given set of data (cross section or time series)?
2. Does the model help the market participants in meeting the relative size of assertions made on future developments?
3. What do the empirical findings imply for the econometric model? Will it eventually have to be modified? Can suggestions actually be made which will influence the functioning and structural organisation of the markets?

In order to handle these empirical questions, a statistical inquiry is needed. Since, as a rule with financial market data, the dynamic characteristics are

the most important, we will mainly concentrate on the time series analysis. First of all, we will introduce the concepts of univariate analysis and then move to the multivariate time series. The interdependence of financial items can be modelled explicitly as a system.

Certain terms, which are often used in time series analysis and in the analysis of financial time series, are introduced in a compact form. We will briefly define them in the next section.

## 11.1 Some Definitions

First we will need to look closer at stochastic processes, the basic object in time series analysis.

**Definition 11.1 (stochastic process)**
*A stochastic process $X_t$, $t \in \mathbb{Z}$, is a family of random variables, defined in a probability space $(\Omega, \mathcal{F}, \mathrm{P})$.*

At a specific time point $t$, $X_t$ is a random variable with a specific density function. Given a specific $\omega \in \Omega$, $X(\omega) = \{X_t(\omega), t \in \mathbb{Z}\}$ is a realisation or a path of the process.

**Definition 11.2 (cdf of a stochastic process)**
*The joint cumulative distribution function (cdf) of a stochastic process $X_t$ is defined as*

$$F_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = \mathrm{P}(X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n).$$

The stochastic process $X_t$ is clearly identified, when the system of its density functions is known. If for any $t_1, \ldots, t_n \in \mathbb{Z}$ the joint distribution function $F_{t_1,\ldots,t_n}(x_1, \ldots, x_n)$ is known, the underlying stochastic process is uniquely determined.

**Definition 11.3 (conditional cdf)**
*The conditional cdf of a stochastic process $X_t$ for any $t_1, \ldots, t_n \in \mathbb{Z}$ with $t_1 < t_2 < \ldots < t_n$ is defined as*

$$F_{t_n|t_{n-1},\ldots,t_1}(x_n \mid x_{n-1}, \ldots, x_1) = \mathrm{P}(X_{t_n} \leq x_n \mid X_{t_{n-1}} = x_{n-1}, \ldots, X_{t_1} = x_1).$$

Next we will define moment functions of the real stochastic process. Here we will assume that the moments exist. If this is not the case, then the corresponding function is not defined.

**Definition 11.4 (Mean function)**
*The mean function $\mu_t$ of a stochastic process $X_t$ is defined as*

$$\mu_t = \mathsf{E}[X_t] = \int_{\mathbb{R}} x dF_t(x). \tag{11.1}$$

In general $\mu_t$ depends on time $t$, as, for example, processes with a seasonal or periodical structure or processes with a deterministic trend.

**Definition 11.5 (Autocovariance function)**
*The autocovariance function of a stochastic process $X$ is defined as*

$$\begin{aligned} \gamma(t,\tau) &= \mathsf{E}[(X_t - \mu_t)(X_{t-\tau} - \mu_{t-\tau})] \\ &= \int_{\mathbb{R}^2} (x_1 - \mu_t)(x_2 - \mu_{t-\tau}) dF_{t,t-\tau}(x_1, x_2) \end{aligned} \tag{11.2}$$

*for $\tau \in \mathbb{Z}$.*

The autocovariance function is symmetric, i.e., $\gamma(t,\tau) = \gamma(t-\tau, -\tau)$. For the special case $\tau = 0$ the result is the variance function $\gamma(t,0) = \mathrm{Var}(X_t)$. In general $\gamma(t,\tau)$ is dependent on $t$ as well as on $\tau$. Below we define the important concept of stationarity, which will simplify the moment functions in many cases.

**Definition 11.6 (Stationary)**
*A stochastic process $X$ is covariance stationary if*

1. *$\mu_t = \mu$, and*
2. *$\gamma(t,\tau) = \gamma_\tau$.*

*A stochastic process $X_t$ is strictly stationary if for any $t_1, \ldots, t_n$ and for all $n, s \in \mathbb{Z}$ it holds that*

$$F_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = F_{t_1+s,\ldots,t_n+s}(x_1, \ldots, x_n).$$

For covariance stationary the term weakly stationary is often used. One should notice, however, that a stochastic process can be strictly stationary without being covariance stationary, namely then, when the variance (or covariance) does not exist. If the first two moment functions exist, then covariance stationary follows from strictly stationary.

**Definition 11.7 (Autocorrelation function (ACF))**
*The autocorrelation function $\rho$ of a covariance stationary stochastic process is defined as*

$$\rho_\tau = \frac{\gamma_\tau}{\gamma_0}.$$

The ACF is normalised on the interval [-1,1] and thus simplifies the interpretation of the autocovariance structure from various stochastic processes. Since the process is required to be covariance stationary, the ACF depends only on one parameter, the lag $\tau$. Often the ACF is plotted as a function of $\tau$, the so called *correlogram*. This is an important graphical instrument to illustrate linear dependency structures of the process.

Next we define two important stochastic processes which build the foundation for further modelling.

**Definition 11.8 (White noise (WN))**
*The stochastic process $X_t$ is white noise if the following holds*

1. $\mu_t = 0$, *and*
2. $\gamma_\tau = \begin{cases} \sigma^2 & \text{when } \tau = 0 \\ 0 & \text{when } \tau \neq 0. \end{cases}$

*SFEtimewr*

If $X_t$ is a process from i.i.d. random values with expectation 0 and finite variance, then it is a white noise. This special case is called *independent white noise*. On the contrary the white noise could have dependent third or higher moments, and in this case it would not be independent.

**Definition 11.9 (Random Walk)**
*The stochastic process $X_t$ follows a* random walk*, if it can be represented as*

$$X_t = c + X_{t-1} + \varepsilon_t$$

*with a constant $c$ and white noise $\varepsilon_t$.*

If $c$ is not zero, then the variables $Z_t = X_t - X_{t-1} = c + \varepsilon_t$ have a non-zero mean. We call it a random walk with a drift (see Section 4.1). In contrast to Section 4.3 we do not require the variables here to be independent. The random walk defined here is the boundary case for an AR(1) process introduced in Example 11.1 as $\alpha \to 1$. When we require, as in Section 4.3, that $\varepsilon_t$ is independent white noise, then we will call $X_t$ a random walk with independent increments. Historically the random walk plays a special role, since at the beginning of the last century it was the first stochastic model to represent the development of stock prices. Even today the random walk is often assumed as an underlying hypothesis. However the applications are rejected in their strongest form with independent increments.

In order to determine the moment functions of a random walk, we will simply assume that the constant $c$ and the initial value $X_0$ are set to zero. Then, through recursive substitutions we will get the representation:

$$X_t = \varepsilon_t + \varepsilon_{t-1} + \ldots + \varepsilon_1.$$

The mean function is simply:

$$\mu_t = \mathsf{E}[X_t] = 0, \tag{11.3}$$

and for the variance function, since there is no correlation of $\varepsilon_t$, we obtain:

$$\mathrm{Var}(X_t) = \mathrm{Var}(\sum_{i=1}^{t} \varepsilon_i) = \sum_{i=1}^{t} \mathrm{Var}(\varepsilon_i) = t\sigma^2. \tag{11.4}$$

The variance of the random walk increases linearly with time. For the autocovariance function the following holds for $\tau < t$:

$$\begin{aligned} \gamma(t,\tau) &= \mathrm{Cov}(X_t, X_{t-\tau}) \\ &= \mathrm{Cov}(\sum_{i=1}^{t} \varepsilon_i, \sum_{j=1}^{t-\tau} \varepsilon_j) \\ &= \sum_{j=1}^{t-\tau} \sum_{i=1}^{t} \mathrm{Cov}(\varepsilon_i, \varepsilon_j) \\ &= \sum_{j=1}^{t-\tau} \sigma^2 = (t-\tau)\sigma^2. \end{aligned}$$

For $\tau < t$ the autocovariance is thus strictly positive. Since the covariance function depends on time $t$ (and not only on the lags $\tau$), the random walk is not covariance stationary. For the autocorrelation function $\rho$ we obtain

$$\rho(t,\tau) = \frac{(t-\tau)\sigma^2}{\sqrt{t\sigma^2(t-\tau)\sigma^2}} = \frac{(t-\tau)}{\sqrt{t(t-\tau)}} = \sqrt{1-\frac{\tau}{t}}.$$

Again $\rho$ depends on $t$ as well as on $\tau$, thus the random walk is not covariance stationary.

As a further illustration we consider a simple, but important stochastic process.

**Example 11.1 (AR(1) Process)**
*The stochastic process $X_t$ follows an autoregressive process of first order, written AR(1) process, if*

$$X_t = c + \alpha X_{t-1} + \varepsilon_t$$

*with a constant parameter $\alpha$, $|\alpha| < 1$. The process $X_t$ can also, through iterative substitutions, be written as*

$$\begin{aligned} X_t &= c(1 + \alpha + \alpha^2 + \ldots + \alpha^{k-1}) \\ &\quad + \alpha^k X_{t-k} + \varepsilon_t + \alpha\varepsilon_{t-1} + \ldots + \alpha^{k-1}\varepsilon_{t-k+1} \\ &= c(\sum_{i=0}^{k-1} \alpha^i) + \alpha^k X_{t-k} + \sum_{i=0}^{k-1} \alpha^i \varepsilon_{t-i} \\ &= c\,\frac{1-\alpha^k}{1-\alpha} + \alpha^k X_{t-k} + \sum_{i=0}^{k-1} \alpha^i \varepsilon_{t-i} \end{aligned}$$

*If $X_{t-k}$ is given for a particular $k$ (for example, the initial value of the process), the characteristics of the process are obviously dependent on this value. This influence disappears, however, over time, since we have assumed that $|\alpha| < 1$ and thus $\alpha^k \to 0$ for $k \to \infty$. For $k \to \infty$ a limit in the sense of squared deviation exists, thus we can write the process $X_t$ as*

$$X_t = c\frac{1}{1-\alpha} + \sum_{i=0}^{\infty} \alpha^i \varepsilon_{t-i}.$$

*For the moment functions we then have*

$$\mu_t = c\frac{1}{1-\alpha},$$

*and*

$$\gamma_\tau = \frac{\sigma^2}{1-\alpha^2}\alpha^\tau.$$

*The ACF is thus simply $\rho_\tau = \alpha^\tau$. For positive $\alpha$ this function is strictly positive, for negative $\alpha$ it alternates around zero. In every case it converges to zero, but with $\alpha = 0.5$, for example, convergence is very fast, and with $\alpha = 0.99$ it is quite slow.* *SFEacfar1*

**Definition 11.10 (Markov Process)**
*A stochastic process has the Markov property if for all* $t \in \mathbb{Z}$ *and* $k \geq 1$

$$F_{t|t-1,\ldots,t-k}(x_t|x_{t-1},\ldots,x_{t-k}) = F_{t|t-1}(x_t|x_{t-1}).$$

In other words, the conditional distribution of a Markov process at a specific point in time is entirely determined by the condition of the system at the previous date. One can also define Markov processes of a higher order, from which the conditional distribution only depends on the finite number of past levels. Two examples of the Markov process of the first order are, the above mentioned random walk with independent variables and the AR(1) process with independent white noise.

**Definition 11.11 (Martingale)**
*The stochastic process* $X_t$ *is a martingale if the following holds*

$$\mathsf{E}[X_t|X_{t-1} = x_{t-1},\ldots,X_{t-k} = x_{t-k}] = x_{t-1}$$

*for every* $k > 0$.

The martingale is also a frequently used instrument in describing prices in financial markets. One should notice that for a martingale process only one statement about the conditional expectation is made, while for a Markov process statements on the entire conditional distribution are made. An example of a martingale is the random walk without a drift. The AR(1) process with $0 < \alpha < 1$ is not a Martingale, since $\mathsf{E}[X_t|x_{t-1},\ldots,x_{t-k}] = c + \alpha x_{t-1}$.

**Definition 11.12 (fair game)**
*The stochastic process* $X_t$ *is a fair game if the following holds*

$$\mathsf{E}[X_t|X_{t-1} = x_{t-1},\ldots,X_{t-k} = x_{t-k}] = 0$$

*for every* $k > 0$.

Sometimes a fair game is also called a martingale difference. If $X_t$ is namely a martingale, then $Z_t = X_t - X_{t-1}$ is a fair game.

**Definition 11.13 (Lag-Operator)**
*The operator* $L$ *moves the process* $X_t$ *back by one unit of time, i.e.,* $LX_t = X_{t-1}$ *and* $L^k X_t = X_{t-k}$. *In addition we define the difference operator* $\Delta$ *as* $\Delta = 1 - L$, *i.e.,* $\Delta X_t = X_t - X_{t-1}$, *and* $\Delta^k = (1-L)^k$.

After these mathematical definitions we arrive at the more econometric definitions, and in particular, at the term return. We start with a time series of prices $P_1, \ldots, P_n$ and are interested in calculating the return between two periods.

**Definition 11.14 (simple return)**
*The simple return $R_t$ is defined as*

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}.$$

Should the average return $R_t(k)$ need to be calculated over $k$ periods, then the geometric mean is taken from the simple gross return, i.e.,

$$R_t(k) = \left( \prod_{j=0}^{k-1} (1 + R_{t-j}) \right)^{1/k} - 1.$$

In general the geometric mean is not equal to the arithmetic mean $k^{-1} \sum_{j=0}^{k-1} R_{t-j}$.

**Definition 11.15 (log return)**
*The log return $r_t$ is defined as*

$$r_t = \ln \frac{P_t}{P_{t-1}} = \ln(1 + R_t).$$

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The log return is defined for the case of continuous compounding. For the average return over several periods we have

$$\begin{aligned} r_t(k) &= \ln\{1 + R_t(k)\} = \frac{1}{k} \ln \prod_{j=0}^{k-1} (1 + R_{t-j}) \\ &= \frac{1}{k} \sum_{j=0}^{k-1} \ln(1 + R_{t-j}) \\ &= \frac{1}{k} \sum_{j=0}^{k-1} r_{t-j}, \end{aligned}$$

i.e., for log returns the arithmetic average return is applied.

For small price changes the difference of the simple return and log return is negligible. According to the Taylor approximation it follows that:

$$\begin{aligned}\ln(1+x) &= \ln(1) + \frac{\partial \ln x}{\partial x}(1)x + \frac{\partial^2 \ln x}{\partial x^2}(1)\frac{x^2}{2!} + \dots \\ &= x - \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .\end{aligned}$$

For $x$ close to zero a first order approximation is sufficient, i.e., $\ln(1+x) \approx x$. As a general rule one could say, that with returns under 10% it does not really matter whether the simple or the log returns are used. This is above all the case when one is studying financial time series with a high frequency, as, for example, with daily values.

## 11.2 Statistical Analysis of German Stock Returns

In this section we describe several classical characteristics of financial time series using daily returns of German stocks from 1974 to 1996. We will concentrate, on the one hand, on the linear, chronological (in)dependence of the returns, and on the other hand, on the distribution characteristics. Table 11.1 displays the summarised descriptive statistics. The autocorrelation of first order is for all stock returns close to zero. The largest positive autocorrelation is with PREUSSAG (0.08), the largest negative autocorrelation is with ALLIANZ (-0.06). The majority of autocorrelations are positive (14 as compared to 6 negative). This is an empirical phenomenon which is also documented for the American market.

While the first order autocorrelation of the returns of all stock returns are all close to zero, the autocorrelations of the squared and absolute returns of all stocks are positive and significantly larger than zero. Obviously there is a linear relationship in the absolute values of the chronologically sequential returns. Since the autocorrelation is positive, it can be concluded, that small (positive or negative) returns are followed by small returns and large returns follow large ones again. In other words, there are quiet periods with small prices changes and turbulent periods with large oscillations. Indeed one can further conclude that these periods are of relatively longer duration, i.e., the autocorrelations of squared returns from mainly very large orders are still positive. These effects have already been examined by Mandelbrot and Fama in the sixties. They can be modelled using, among others, the ARCH models studied in Chapter 13. Furthermore we will consider estimates for the skewness and kurtosis. Whereas the skewness in most cases is close to zero and is sometimes positive, sometimes negative, the kurtosis is in

every case significantly larger than 3. The smallest estimated kurtosis is by THYSSEN ($\widehat{\text{Kurt}} = 6.1$), the largest by ALLIANZ ($\widehat{\text{Kurt}} = 32.4$). Under the null hypothesis of the normal distribution, the estimates in (3.2) and (3.3) are independent and asymptotically normally distributed with

| | $\rho_1(r_t)$ | $\rho_1(r_t^2)$ | $\rho_1(\lvert r_t\rvert)$ | $S$ | $K$ | $BJ$ |
|---|---|---|---|---|---|---|
| ALLIANZ | -0.0632 | 0.3699 | 0.3349 | 0.0781 | 32.409 | 207116.0 |
| BASF | -0.0280 | 0.2461 | 0.2284 | -0.1727 | 8.658 | 7693.5 |
| BAYER | -0.0333 | 0.3356 | 0.2487 | 0.0499 | 9.604 | 10447.0 |
| BMW | -0.0134 | 0.3449 | 0.2560 | -0.0107 | 17.029 | 47128.0 |
| COMMERZBANK | 0.0483 | 0.1310 | 0.2141 | -0.2449 | 10.033 | 11902.0 |
| DAIMLER | -0.0273 | 0.4050 | 0.3195 | 0.0381 | 26.673 | 134201.0 |
| DEUTSCHE BANK | 0.0304 | 0.2881 | 0.2408 | -0.3099 | 13.773 | 27881.0 |
| DEGUSSA | 0.0250 | 0.3149 | 0.2349 | -0.3949 | 19.127 | 62427.0 |
| DRESDNER | 0.0636 | 0.1846 | 0.2214 | 0.1223 | 8.829 | 8150.2 |
| HOECHST | 0.0118 | 0.2028 | 0.1977 | -0.1205 | 9.988 | 11708.0 |
| KARSTADT | 0.0060 | 0.2963 | 0.1964 | -0.4042 | 20.436 | 72958.0 |
| LINDE | -0.0340 | 0.1907 | 0.2308 | -0.2433 | 14.565 | 32086.0 |
| MAN | 0.0280 | 0.2824 | 0.2507 | -0.5911 | 18.034 | 54454.0 |
| MANNESMANN | 0.0582 | 0.1737 | 0.2048 | -0.2702 | 13.692 | 27442.0 |
| PREUSSAG | 0.0827 | 0.1419 | 0.1932 | 0.1386 | 10.341 | 12923.0 |
| RWE | 0.0408 | 0.1642 | 0.2385 | -0.1926 | 16.727 | 45154.0 |
| SCHERING | 0.0696 | 0.2493 | 0.2217 | -0.0359 | 9.577 | 10360.0 |
| SIEMENS | 0.0648 | 0.1575 | 0.1803 | -0.5474 | 10.306 | 13070.0 |
| THYSSEN | 0.0426 | 0.1590 | 0.1553 | -0.0501 | 6.103 | 2308.0 |
| VOLKSWAGEN | 0.0596 | 0.1890 | 0.1687 | -0.3275 | 10.235 | 12637.0 |

Table 11.1: First order autocorrelation of the returns $\rho_1(r_t)$, the squared returns $\rho_1(r_t^2)$ and the absolute returns $\rho_1(|r_t|)$ as well as skewness ($S$), kurtosis ($K$) and the Bera-Jarque test statistic ($BJ$) for the daily returns of German stocks 1974-1996.

SFEReturns

$$\sqrt{n}\hat{S} \xrightarrow{\mathcal{L}} \text{N}(0,6)$$

and

$$\sqrt{n}(\widehat{\text{Kurt}} - 3) \xrightarrow{\mathcal{L}} \text{N}(0,24).$$

From this the combined test of the normal distribution from Bera and Jarque ($BJ$) can be derived:

$$BJ = n\left(\frac{\hat{S}^2}{6} + \frac{(\widehat{\text{Kurt}} - 3)^2}{24}\right).$$

$BJ$ is asymptotically $\chi^2$ distribution with two degrees of freedom. The last column in Table 11.1 shows, that in all cases the normal distribution hypothesis is clearly rejected by a significance level of 1% (critical value 9.21). This

is above all caused by the value of the kurtosis. Typically in financial time series, the kurtosis is significantly larger than 3, which is caused by the frequent appearance of outliers. Furthermore, there are more frequent appearances of very small returns than what one would expect under the normal distribution hypothesis. One says that the empirical distribution of the returns is *leptokurtic*, which means that the distribution is more mass around the centre and in the tails than the normal distribution. The opposite, a weaker asymmetry or *platykurtic* distribution rarely appears in financial markets.

## 11.3 Expectations and Efficient Markets

Market efficiency is a very general concept in economic theory. A market is called efficient if at every point in time all relevant information is completely reflected in the price of the traded object. This general definition must be defined more concretely, in order to say what "completely reflected" means. To this end we require the concept of *rational expectations*. In general one speaks of rational expectations when by the forecast of a stochastic process $P_t$ all relative and available information $\mathcal{F}_{t-1}$ (see Definition 5.1) is 'optimally' used. Optimal means that the mean squared error of the forecast is minimised. This is the case when the conditional expectation (see Section 3.5) $\mathsf{E}[P_t|\mathcal{F}_{t-1}]$ is used as the forecast.

**Theorem 11.1**
*For every $h > 0$ using the conditional expectation $\mathsf{E}[P_{t+h} \mid \mathcal{F}_t]$ as a forecast, $P^*_{t+h|t}$ minimises the mean squared error $\mathsf{E}[(P_{t+h} - P^*_{t+h|t})^2]$ given all relevant information $\mathcal{F}_t$ at time $t$.*

**Proof:**
Given any forecast $P^*_{t+h|t}$, that can be written as a (in general nonlinear) function of the random variables at time $t$, which determines the information set $\mathcal{F}_t$, then the mean squared error can be written as:

$$\begin{aligned}
\mathsf{E}[(P_{t+h} - P^*_{t+h|t})^2] &= \mathsf{E}[(P_{t+h} - \mathsf{E}[P_{t+h}|\mathcal{F}_t] + \mathsf{E}[P_{t+h}|\mathcal{F}_t] - P^*_{t+h|t})^2] \\
&= \mathsf{E}[(P_{t+h} - \mathsf{E}[P_{t+h}|\mathcal{F}_t])^2] \\
&\quad + \mathsf{E}[(\mathsf{E}[P_{t+h}|\mathcal{F}_t] - P^*_{t+h|t})^2], \qquad (11.5)
\end{aligned}$$

since the cross product is equal to zero:

$$\begin{aligned} 2\mathsf{E}[(P_{t+h} - \mathsf{E}[P_{t+h}|\mathcal{F}_t])\,(\mathsf{E}[P_{t+h}|\mathcal{F}_t] - P^*_{t+h|t})] &= \\ 2\mathsf{E}[\mathsf{E}[P_{t+h} - \mathsf{E}[P_{t+h}|\mathcal{F}_t]|\mathcal{F}_t](\mathsf{E}[P_{t+h}|\mathcal{F}_t] - P^*_{t+h|t})] &= \\ 2\mathsf{E}[0 \cdot (\mathsf{E}[P_{t+h}|\mathcal{F}_t] - P^*_{t+h|t})] &= 0. \end{aligned}$$

The second term on the right hand side of (11.5) is non-negative and is equal to zero when $\mathsf{E}[P_{t+h}|\mathcal{F}_t] = P^*_{t+h|t}$.

□

Not all economic variables have sufficient information available to estimate $\mathsf{E}[P_t \mid \mathcal{F}_{t-1}]$. This has to do with the type of underlying process that determines $P_t$ and the relative level of the necessary information for the forecast. In order to shed light upon this conceptual problem, hypotheses have been developed in the macro-economic theory, which do not require the use of mathematical expectations $\mathsf{E}[P_t \mid \mathcal{F}_{t-1}]$. The hypothesis on *adaptive expectations* assumes, for instance, that the forecast at time $t-1$ of $P_t$, $\mathsf{E}^a_{t-1}[P_t]$, is generated by the following mechanism:

$$\mathsf{E}^a_{t-1}[P_t] - \mathsf{E}^a_{t-2}[P_{t-1}] = \theta(P_{t-1} - \mathsf{E}^a_{t-2}[P_{t-1}]) \tag{11.6}$$

with a constant parameter $\theta,\ 0 < \theta < 1$. Changes in the forecast result from the last forecast error weighted by $\theta$.

**Theorem 11.2**
*The adaptive expectation in (11.6) is optimal in the sense of the mean squared error exactly when $P_t$ follows the process*

$$P_t = P_{t-1} + \varepsilon_t - (1-\theta)\varepsilon_{t-1} \tag{11.7}$$

*where $\varepsilon_t$ is white noise.*

**Proof:**
With the Lag-Operator $L$ (see Definition 11.13), (11.6) can be represented as

$$\{1 - (1-\theta)L\}\,\mathsf{E}^a_{t-1}[P_t] = \theta P_{t-1}.$$

Since $0 < \theta < 1$ and $\{1-(1-\theta)z\}^{-1} = \sum_{i=0}^{\infty}(1-\theta)^i z^i$ this can be written as

$$\mathsf{E}^a_{t-1}[P_t] = \theta \sum_{i=0}^{\infty}(1-\theta)^i P_{t-i-1}.$$

The process (11.7) can be rewritten as

$$\{1 - (1-\theta)L\}\,\varepsilon_t = P_t - P_{t-1}$$

and

$$\begin{aligned}\varepsilon_t &= \sum_{j=0}^{\infty}(1-\theta)^j(P_{t-j} - P_{t-j-1}) \\ &= P_t - \theta\sum_{j=0}^{\infty}(1-\theta)^j P_{t-j-1},\end{aligned}$$

so that $P_t - \mathsf{E}^a_{t-1}[P_t]$ is white noise. Thus $\mathsf{E}^a_{t-1}[P_t]$ is the best forecast for $P_t$ in the sense of the mean squared error.

□

The process (11.7) is also referred to as the integrated autoregressive moving average process (ARIMA) of order (0,1,1). The family of ARIMA models will be discussed in more detail in Chapter 12. In general exogenous factors, for example, supply shocks, could also be involved in determining the equilibrium prices. In this case adaptive expectations would be suboptimal. If $X_t$ is the stochastic exogenous factor and $\mathcal{F}_t$ is a family of results which are determined from the observations $\{p_t, p_{t-1}, \ldots, x_t, x_{t-1}, \ldots\}$ available at time $t$, then the optimal process $\mathsf{E}[P_t \mid \mathcal{F}_{t-1}]$ is in general a function of $\{p_t, p_{t-1}, \ldots\}$ and of $\{x_t, x_{t-1}, \ldots\}$. Special cases do exist in which adaptive expectations coincide with rational expectations, for example, in a linear supply/demand system with $X_t$ as an exogenous shock that follows a random walk. If $X_t$ is instead an AR(1) process, then forecasts with adaptive expectations have a larger mean squared error than forecasts with rational expectations. If the factor $X_t$ is *common knowledge*, i.e., available to the public, then rational expectations in this example would mean that the price would be optimally forecasted by using this information.

However, when the factor $X_t$ is not observable for everyone, in principle the uninformed agent could *learn* from the prices offered by the informed agent. This means that through observation of prices they could obtain information on the status of $\omega$, above and beyond what is in their private information set $F_i$. Here it is assumed that the information function of prices is correctly interpreted.

In order to illustrate what role the price plays in forming expectations, imagine purchasing a bottle of wine. In the store there are three bottles to choose with the prices EUR 300, EUR 30 and EUR 3. Since the bottle for EUR

| Type | P1 | P2 |
|---|---|---|
| I | 300 | 50 |
| II | 50 | 300 |
| III | 150 | 250 |

Table 11.2: Payments in periods P1 and P2 according to type of investor

300 exceeds the budget, only two bottles for EUR 3 and EUR 30 are considered. Now assume that someone who is not a wine expert could not evaluate the quality of the wine from the information on the label. Since one is often pressed for time, collecting information from other sources is time consuming, so what remains is the information implied by the price. Assume further that one has learned through previous shopping experiences that the more expensive wine tends to be better than the cheaper wine. Thus one constructs a function of the price with respect to the quality, i.e., how good the wine is. One would choose the wine for EUR 30 if the better quality and more expensive wine was valued more in the utility function than the price advantage of the cheaper wine. The buyer behaved *rationally*, since he optimised his decision (here maximising his utility function) with the help of the available information and the price function, assuming that the function was right.

In addition let's take a look at another example of an experimental market which is taken from the literature. We have a security that is traded in two periods P1 and P2 and in each period it pays various dividends according to the type of investor. The trading system is an auction in which at the time of an offer both bid and ask prices are verbally given. There are three types of investors and from each type there are three investors, i.e., a total of nine investors can trade the security, among other instruments. Each investor has an initial capital of 10 000 Franks (1 'Frank' = 0.002 USD) and two securities. The initial capital of 10 000 Franks must be paid back at the end of the second period. Every profit which results from trading the security may be kept. When the investor is in possession of the security at the end of P1 or P2, he will receive the dividend with respect to what type of investor he is. Table 11.2 displays information on the dividend payments.

Every investor knows only his own dividend payment, no one else. The question is, whether, and if so how fast, the investors 'learn' about the pricing structure, i.e., gain information on the value of the security to the other investors. There are two underlying hypotheses:

1. Investors tell each other through their bids about their individual dividends only in P1 and P2 ('naive behaviour').

2. The investors draw conclusions through the observed price on the value of the security for the other investors and use this information in their own bids ('rational behaviour').

Since the experiment is over after the period P2, only the individual dividend payments of each investor are of interest, so that in P2 both hypotheses coincide: The equilibrium price is 300 Franks, since type II is just willing to buy at this price and there is competition among the type II investors. At the beginning of P1, before any trading begins, each investor has information only on his own dividends, so that at first one applies naive behaviour: type I and type II would offer a maximum of 350, type III would offer a maximum of 400, thus the equilibrium price according to the hypothesis of naive behaviour is 400 Franks. This hypothesis performed well in empirical experiments. When the experiment is repeated with the same dividend matrix, the investors can learn through the prices of the previous experiment, which value the security has for the other types of investors in P2. In particular under the hypothesis of rational behaviour, type I could learn that the equilibrium price in P2 is higher than what his own dividend would be, thus he could sell the security at a higher price. The equilibrium price in P1 is under the rational hypothesis 600 Franks. Type I buys at the price in P1 and sells in P2 to type II at a price of 300.

In repeated experiments it was discovered that the participants actually tended from naive behaviour to rational behaviour, although the transition did not occur immediately after the first experiment, it was gradual and took about 8 repetitions. Other experiments were run, including a *forward and futures market* in which in the first period P1 the price of the security in P2 could already be determined. Here it was shown that through the immediate transparency of the security's value in future periods the transition to rational expectations equilibrium was much quicker.

The observed market price is created through the interaction of various supplies and demands an aggregation of the individual heterogenous information sets. Assume that the price at time $t$ is a function of the state of the economy, the *price function* $p_t(\omega)$, $\omega \in \Omega$. Below we define an equilibrium with rational expectations.

**Definition 11.16 (RE-Equilibrium)**
*An equilibrium at $t$ with rational expectations (RE-equilibrium) is an equilibrium in which every agent $i$ optimises their objective function given the information set $\mathcal{F}_{i,t}$ and the price function $p_t(\omega)$.*

Definition 11.16 assumes in particular that every agent includes the information function of the prices correctly in his objective function.

The concept of *efficient markets* is closely related to the concept of rational expectations. According to the original and general definition, a market is efficient if at every point in time all relevant information is reflected in the price. This means, for example, that new information is immediately incorporated into the price. Below we define efficient markets with respect to an information set $\mathcal{G}$.

**Definition 11.17 (Efficient Markets)**
*A market is efficient with respect to $\mathcal{G} = (G_t)$, $t \in \mathbb{N}$, $G_t \subset \mathcal{F}_t$, if at every time $t$ the market is in RE-equilibrium with the price function $p_t(\omega)$ and if for every agent $i$ and every time $t$ the following holds*

$$G_t \subset \{\mathcal{F}_{i,t} \cup p_t(\omega)\}.$$

Typically three cases are identified as weak, semi-strong and strong efficiency.

1. The market is *weak efficient*, when efficiency refers only to historical prices, i.e., the set $\mathcal{G} = (G_t)$,
   $G_t = \{p_t, p_{t-1}, p_{t-2}, \ldots\}$. This is, for example, achieved when for all $i$ it holds that $\{p_t, p_{t-1}, p_{t-2}, \ldots\} \subset F_{i,t}$, that is when the historical prices are contained in every private information set.
2. The market is *semi-strong efficient*, when efficiency refers to the set $\mathcal{G} = (G_t)$, $(\cap_i F_{i,t}) \subset G_t \subset (\cup_i F_{i,t})$, which includes all publicly available information.
3. The market is *strong efficient*, when efficiency refers to the set $\mathcal{G} = (G_t), G_t = \cup_i F_{i,t}$, i.e., when all information (public and private) is reflected in the price function. In this case one speaks of a *fully revealing* RE-equilibrium.

An equivalent definition says that under efficient markets no abnormal returns can be achieved. In order to test it one must first determine what a 'normal' return is, i.e., one must define an econometric model. Efficient markets can then be tested only with respect to this model. If this combined hypothesis is rejected, it could be that markets are inefficient or that the econometric model is inadequate.

The following is a brief summary of the typical econometric models that have been proposed for financial data. For each of the most interesting financial instruments – stocks, exchange rates, interest rates and options – a corresponding theory will be presented; each is considered to be classic theory in its respective area. In later chapters we will refer back to these theories when discussing empirically motivated expansions.

## 11.4 Econometric Models: A Brief Summary

### 11.4.1 Stock Prices: the CAPM

The *capital asset pricing model* (CAPM), developed independently by various authors in the sixties, is a classical equilibrium model for the valuation of risky securities (stocks). It is based on the following assumptions:

1. Homogenous information among the market participants exists. This belief can be weakened by assuming that under homogenous information a rational equilibrium is *fully revealing* (see the strong version of Definition 11.17).
2. The market has no friction, i.e., there are no transaction costs, no taxes, no restrictions on short selling or on the divisibility of stocks.
3. There is complete competition.
4. There are no arbitrage opportunities.
5. There are a finite number of stocks ($K$) and risk free security with return $r$.
6. Every investor has a strictly concave utility function as a function of risky future cash flow. This means that every investor is risk averse.
7. Every investor maximises their expected utility, which is dependent only on the expectation and variance of risky future cash flow. This is the crucial assumption of the CAPM. Sufficient conditions for this $(\mu - \sigma)$-criterion are either of the following:
   a) Every investor has a quadratic utility function.
   b) The stock returns are normally distributed.

Below $X_{i,t}$ and $\alpha_{i,t}$ represent the price and the number of $i$-th stock supplied in equilibrium at time $t$. We define the *market portfolio* $X_{m,t}$ as

$$X_{m,t} = \sum_{i=1}^{K} \alpha_{i,t} X_{i,t}. \tag{11.8}$$

The relative weight $w_{i,t}$ of the $i$-th stock in this portfolio is as follows:

$$w_{i,t} = \frac{\alpha_{i,t} X_{i,t}}{\sum_k \alpha_{k,t} X_{k,t}}.$$

Most of the well known stock indices are value weighted indices, nevertheless often only the largest stocks on the market are included in the index (DAX, for example, contains only the 30 largest stocks). As in Definition 11.15,

we define the stock return as $R_{i,t} = \ln(X_{i,t}/X_{i,t-1})$ and the market return as $R_{m,t} = \ln(X_{m,t}/X_{m,t-1})$. We assume that the underlying process of the return is covariance stationary. In equilibrium according to the CAPM it holds for every stock $i$ that

$$\mathsf{E}[R_{i,t}] = r + \beta_i(\mathsf{E}[R_{m,t}] - r), \tag{11.9}$$

with the 'beta' factor

$$\beta_i = \frac{\mathrm{Cov}(R_{i,t}, R_{m,t})}{\mathrm{Var}(R_{m,t})}.$$

Equation (11.9) says that in equilibrium the expected return of the $i$-th stock is comprised of two components: the return of risk free security and a risk premium which is specifically determined for each stock through the beta factor. Stocks that are positively correlated with the market have a positive risk premium. The larger the correlation of a stock with the market portfolio is, the larger is the premium in CAPM for portfolio risk.

Since the CAPM can be derived using theories on utilities, it is sometimes described as a demand oriented equilibrium model. In contrast to this there are other models that explain the stock returns in terms of various aggregate variables, so called *factors*, and are referred to as being supply oriented. In Section 13.4 we will relax the assumptions of time constant variance and covariance implicit in equation (11.9).

### 11.4.2 Exchange Rate: Theory of the Interest Rate Parity

For stocks one can find a large variety of econometric models and for exchange rates there are even more. There are two standard and quite simple theories. However they are not sufficient to explain the considerable price movements in currency markets, especially in the short-run. The *purchasing power parity* (PPP) assumes that identical goods in different countries must have the same relative price, i.e., a relative price given in units of currency. It has been empirically determined that in the long-run this theory describes reality well, but in the short-term price movements could not be explained. The second simple theory, the *theory of interest rate parity*, performs better as capital flows faster than goods. The difference in interest rates can thus resemble the exchange of capital in other currencies, as does the exchange rate. The theory of interest rate parity assumes that domestic and foreign securities are perfect substitutes with respect to duration and risk structure.

Assume that along with forward and futures markets currency can be traded over time. The *spot price* is calculated by $W_t^K$, the forward and future price by $W_t^T$, each is given in units of the foreign currency, i.e., EUR/USD. An

internationally acting investor has two choices. Either he holds a domestic capital investment with the domestic interest rate $r_t^i$ or he chooses a foreign investment with the foreign interest rate $r_t^a$. If he chooses the foreign investment, he must first exchange his capital into foreign currency at the spot price and at the end, exchange back again. The uncertainty about the future developments of the exchange rate can be avoided by purchasing a forward or future contract. In this case the return on the foreign investment is $(1/W_t^K)(1+r_t^a)W_t^T - 1$. If this return is not equal to the domestic interest rate, then an equilibrium has not been reached. Through immediate price adjustments the interest rate arbitrage disappeares and then equilibrium is reached. Thus in equilibrium it must hold that

$$\frac{W_t^T}{W_t^K} = \frac{1+r_t^i}{1+r_t^a}, \tag{11.10}$$

i.e., the relationship between forward and future markets and spot markets corresponds exactly to the relationship between domestic and foreign gross interest rates. The relationship in (11.10) is also called the *covered interest rate parity*, since at the time of investment it deals with risk free exchange and interest rates.

In addition to the interest rate arbitrageur, there are the so-called forward and future speculators that compare the expected future exchange rate with the forward and future price and the corresponding risk of purchasing (selling) currency below or above the equilibrium. Consider a simple case where forward and future speculators are risk neutral. Then in equilibrium the expected exchange rate is equal to the forward and future price, i.e.,

$$W_t^T = \mathsf{E}[W_{t+1}^K \mid \mathcal{F}_t], \tag{11.11}$$

with the information set $\mathcal{F}_t$ which contains all relevant and available information. Here we assume that the speculators have rational expectations, i.e., the true underlying process is known and is used to build the optimal forecast by the speculators. This can also be written as the relationship:

$$W_{t+1}^K = \mathsf{E}[W_{t+1}^K \mid \mathcal{F}_t] + \varepsilon_{t+1} \tag{11.12}$$

which says that the deviations of the speculator's forecast $\mathsf{E}[W_{t+1}^K \mid \mathcal{F}_t]$ from the realised exchange rates is white noise $\varepsilon_t$ (see Definition 11.8). The market is inefficient when the speculators actually are risk neutral and $\varepsilon_t$ is not white noise. In this case the set $\mathcal{F}_t$ does not reflect all of the relevant information in the expectations of the speculators - they do not have rational expectations. In order to test for market efficiency (that is, in order to test whether $\varepsilon_t$ is white noise) we first need a model for $\mathsf{E}[W_{t+1}^K \mid \mathcal{F}_t]$. This can be formulated from (11.11) and (11.10).

Substituting (11.11) into (11.10) we obtain the so called *uncovered interest rate parity*,

$$\frac{\mathsf{E}[W_{t+1}^K \mid \mathcal{F}_t]}{W_t^K} = \frac{1 + r_t^i}{1 + r_t^a}. \tag{11.13}$$

This interest rate parity is risky because future exchange rates are uncertain and enter the relationship as expectations.

Together with (11.12) the uncovered interest rate parity (11.13) implies that the following holds

$$W_{t+1}^K = \frac{1 + r_t^i}{1 + r_t^a} W_t^K + \varepsilon_{t+1}. \tag{11.14}$$

When the difference in the long-term interest rates is zero on average, then (11.14) is a random walk (see Definition 11.9). The random walk is the first model to describe exchange rates.

It should be emphasised that the derivation of this simple model occurred under the assumption of risk neutrality of the speculators. In the case of risk aversion, a risk premium must be included. If, for example, we want to test the efficiency of the currency markets, we could then test the combined hypothesis of efficiency and uncovered interest rate parity using risk neutrality. A rejection of this hypothesis indicates market inefficiency or that the interest rate parity model is a poor model for currency markets.

### 11.4.3 Term Structure: The Cox-Ingersoll-Ross Model

Term structure models are applied to model the chronological development of bond returns with respect to time to maturity. The classical starting point is to identify one or more factors which are believed to determine the term structure. Through specification of the dynamic structure and using specific expectation hypotheses, an explicit solution can be obtained for the returns.

As a typical example we briefly introduce the Cox, Ingersoll and Ross (CIR) model, which has already been mentioned in Section 10.2. The price of a *Zero Coupon Bond* with a nominal value of 1 EUR is given by $P_T(t)$ at time $t$, i.e., a security with no dividend payments that pays exactly one EUR at maturity date $T$. The log return of the zero coupon bond is given by $Y_T(t)$. We assume that continuous compounding holds. The process $Y_T(t)$ is frequently referred to as the *yield to maturity* . The relationship between the price and the return of the zero coupon bond is:

$$P_T(t) = \exp\{-Y_T(t)\tau\}$$

with the remaining time to maturity $\tau = T - t$. This can be easily seen from the definition of a log return (Definition 11.15). For very short time intervals the *short rate* $r(t)$ is defined as:

$$r(t) = \lim_{T \to t} Y_T(t).$$

In practice the short rate corresponds to the spot rate, i.e., the interest rate for the shortest possible investment (see Section 10.2). Consider, intuitively, the choice between an investment in a zero bond with the return $Y_T(t)$ and repeatedly investing at a (risky) short-term interest rate in future periods. An important expectation hypothesis says that the following holds

$$P_T(t) = \mathsf{E}\left[\exp(-\int_t^T r(s)ds)|\mathcal{F}_t\right] \tag{11.15}$$

(also see equation (10.1) for variable but deterministic interest). The short rate is frequently seen as the most important predicting factor of the term structure. As the CIR model, most one factor models use the short rate as a factor. The CIR model specifies the dynamic of the short rate as a continuous stochastic process

$$dr(t) = a\{b - r(t)\}dt + \sigma\sqrt{r(t)}dW_t \tag{11.16}$$

with a Wiener process $W_t$ and constant parameters $a, b$ and $\sigma$ - see also Section 10.2. The process (11.16) has a so called *mean reversion* behaviour, i.e., once deviations from the stationary mean $b$ occurs, the process is brought back to the mean value again through a positive $a$. The volatility, written as $\sigma\sqrt{r(t)}$, is larger whenever the interest level is higher, which can also be shown empirically.

Since in the equation (11.16) $r(t)$ is specified as a Markov process, $P_T(t)$ is, as a consequence of equation (11.15), a function of the actual short rate, i.e.,

$$P_T(t) = V\{r(t), t\}.$$

With Itô's lemma (5.12) and (10.7) we obtain from (11.16) the differential equation

$$a(b - r)\frac{\partial V(r,t)}{\partial r} + \frac{1}{2}\sigma^2 r\frac{\partial^2 V(r,t)}{\partial r^2} + \frac{\partial V(r,t)}{\partial t} - rV(r,t) = 0.$$

With the bounding constraint $V(r,T) = P_T(T) = 1$ the following solution is obtained:

$$P_T(t) = V\{r(t), t\} = \exp\{A(T-t) + B(T-t)r(t)\}, \tag{11.17}$$

where (see Section 10.4)

$$\begin{aligned} A(\tau) &= \frac{2ab}{\sigma^2} \ln \frac{2\psi \exp\{(a+\psi)\tau/2\}}{g(\tau)}, \\ B(\tau) &= \frac{2\{1-\exp(\psi\tau)\}}{g(\tau)} \\ \psi &= \sqrt{a^2+2\sigma^2} \\ g(\tau) &= 2\psi + (a+\psi)\{\exp(\psi\tau)-1\}. \end{aligned}$$

For increasing time periods $T-t$ the term structure curve $Y_T(t)$ converges to the value

$$Y_{lim} = \frac{2ab}{\psi+a}.$$

If the short-term interest lies above $b$, then the term structure decreases, see Figure 11.1; if it lies below $Y_{lim}$, then the term structure increases, see Figure 11.2. If the short-term interest rate lies between $b$ and $Y_{lim}$, then the curve could first rise and then fall.

### 11.4.4 Options: The Black-Scholes Model

Since we have thoroughly covered the Black-Scholes model on option pricing in the first part of this book, only a brief summary of the model will be covered here. Options are not only theoretically interesting for financial markets, but also from an empirical point of view. Just recently there have been indications of a systematic deviation of actual market prices from the Black-Scholes prices. These deviations will be discussed in more detail in later chapters, specifically in dealing with ARCH models.

As an example let's consider a European call option on a stock which receives no dividends in the considered time periods and has the spot price $S_t$ at time $t$. $C(S,t)$ is the option price at time $t$, when the actual price is $S_t = S$. The payoff at the time to maturity $T$ is $C(S_T,T) = \max(0, S_T - K)$, where $K$ is the strike price. The option price is determined from general no arbitrage conditions as

$$C(S_t,t) = \mathsf{E}[e^{-r\tau} C(S_T,T)|\mathcal{F}_t],$$

where expectations are built on an appropriate risk neutral distribution - see also (6.23). $r$ is the fixed riskless interest rate.

Special results can only be derived when the dynamics of the stock prices are known. The assumptions made by Black and Scholes are that the stock

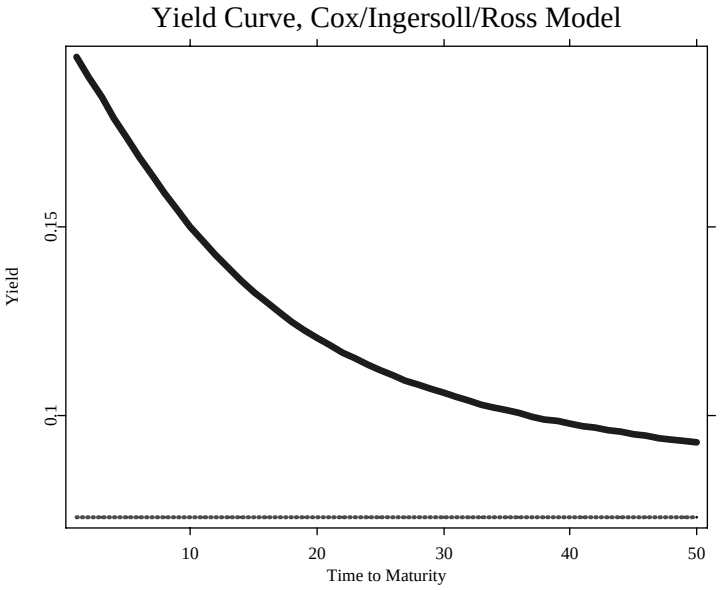


Figure 11.1: Term structure curve according to the Cox-Ingersoll-Ross model with a short rate of $r_t$=0.2, $a = b = \sigma = 0.1$ and $Y_{lim} = 0.073$ (dotted line). SFEcir

prices $S_t$ are geometric Brownian motion, i.e.,

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{11.18}$$

The option price $C(S,t)$ thus satisfies the Black-Scholes differential equation (6.3) as a function of time and stock prices

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} = rC$$

Black and Scholes derive the following solutions (see Section 6.2):

$$C(S,t) = S\Phi(y + \sigma\sqrt{\tau}) - e^{-r\tau}K\Phi(y), \tag{11.19}$$

where $\tau = T - t$ is the time to maturity for the option and $y$ is an abbreviation for

$$y = \frac{\ln \frac{S}{K} + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

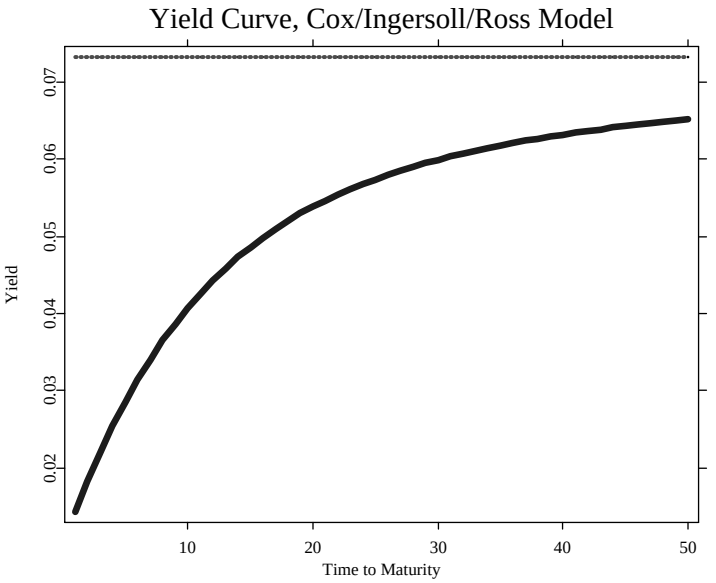


Figure 11.2: Term structure curve according to the Cox-Ingersoll-Ross model with a short rate of $r_t$=0.01, $a = b = \sigma = 0.1$ and $Y_{lim} = 0.073$ (dotted line). SFEcir

### 11.4.5 The Market Price of Risk

In a risk neutral world the market price of risk, see Section 10.3, is equal to zero. In the following section, we will consider the market price of risk and derive once again the Black-Scholes formula. To do this we will consider derivatives of financial instruments that are determined by a single random process $\theta_t$. We will assume that the process $\theta_t$ is geometric Brownian motion:

$$d\theta_t = m\theta_t dt + s\theta_t dW_t \; . \tag{11.20}$$

The variable $\theta_t$ does not necessarily represent a financial value. It could be the state of the market, a measure of the popularity of a politician or the frequency of an ad-hoc announcement at time $t$. Assume that $V_{1t}$ and $V_{2t}$ are the prices for two derivatives of financial instruments that depend only on $\theta_t$ and $t$. As a simplification, no payments are allowed during the observed time period. This process $V_{jt} = V_j(\theta, t)$, $j = 1, 2$ also follows the schema (11.20)

with the *same* Wiener process $W_t$

$$dV_{jt} = \mu_{jt}V_{jt}dt + \sigma_{jt}V_{jt}dW_t, \qquad j = 1, 2 \tag{11.21}$$

where $\mu_{jt}, \sigma_{jt}$ could be functions of $\theta_t$ and $t$. The random process $W_t$ in (11.20) and (11.21) is always the same since we assume that this is the only source that creates uncertainty.

The observation of (11.21) in discrete time leads to:

$$\Delta V_{1t} = \mu_{1t}V_{1t}\Delta t + \sigma_{1t}V_{1t}\Delta W_t \tag{11.22}$$
$$\Delta V_{2t} = \mu_{2t}V_{2t}\Delta t + \sigma_{2t}V_{2t}\Delta W_t \tag{11.23}$$

We could "eliminate the random variable $\Delta W_t$" by constructing a risk free portfolio that continually changes. To do this we take $\sigma_{2t}V_{2t}$ units of the first instrument and $-\sigma_{1t}V_{1t}$ of the second instrument, i.e., we short sell the second instrument. Letting $\Pi_t$ be the total value of the portfolio at time $t$ we have:

$$\Pi_t = (\sigma_{2t}V_{2t})V_{1t} - (\sigma_{1t}V_{1t})V_{2t} \tag{11.24}$$

and

$$\Delta\Pi_t = (\sigma_{2t}V_{2t})\Delta V_{1t} - (\sigma_{1t}V_{1t})\Delta V_{2t} \tag{11.25}$$

Substituting in (11.22) and (11.23) we have:

$$\Delta\Pi_t = (\mu_{1t}\sigma_{2t}V_{1t}V_{2t} - \mu_{2t}\sigma_{1t}V_{1t}V_{2t})\Delta t. \tag{11.26}$$

This portfolio should be risk free, thus in time period $\Delta t$ it must produce the risk free profit $r\Delta t$:

$$\frac{\Delta\Pi_t}{\Pi_t} = r\Delta t. \tag{11.27}$$

Substituting (11.24) and(11.26) into this equation produces:

$$\begin{aligned}(\mu_{1t}\sigma_{2t}V_{1t}V_{2t} - \mu_{2t}\sigma_{1t}V_{1t}V_{2t})\Delta t &= (\sigma_{2t}V_{1t}V_{2t} - \sigma_{1t}V_{1t}V_{2t})r\Delta t \\ \mu_{1t}\sigma_{2t} - \mu_{2t}\sigma_{1t} &= r\sigma_{2t} - r\sigma_{1t} \\ \frac{\mu_{1t} - r}{\sigma_{1t}} &= \frac{\mu_{2t} - r}{\sigma_{2t}}\end{aligned}$$

Equating this as in (10.3) to $\lambda_t$ we see that the price $V_t$ of a derivative instrument, an instrument that depends only on $\theta_t$ and $t$, follows the dynamics

$$dV_t = \mu_t V_t dt + \sigma_t V_t dW_t, \tag{11.28}$$

the value

$$\lambda_t = \frac{\mu_t - r}{\sigma_t} = \frac{\mu(\theta_t, t) - r}{\sigma(\theta_t, t)} \tag{11.29}$$

represents the *market price of risk* . This market price of risk can depend on $\theta_t$ (using $\mu_t, \sigma_t$), but not on the actual price of the instrument $V_t$! We can rewrite the equation (11.29) as:

$$\mu_t - r = \lambda_t \sigma_t \tag{11.30}$$

Furthermore we can interpret $\sigma_t$, which in this interpretation can also be negative, as the level of the $\theta_t$-risk in $V_t$. Equation (11.30) has strong ties to the CAPM model, which we discussed in Section (11.4.1) - for further details see also Hafner and Herwartz (1998).

**Example 11.2** *Assume that there are two objects, both are dependent on a 90 day interest rate. The first instrument has an expected return of* 6% *per year and a volatility of* 20% *per year. For the second instrument a volatility of* 30% *per year is assumed. Furthermore,* $r = 3\%$ *per year. The market price of risk for the first instrument according to (11.29) is:*

$$\frac{0.06 - 0.03}{0.2} = 0.15 \tag{11.31}$$

*By substituting into equation (11.30) for the second object we obtain:*

$$0.03 + 0.15 \cdot 0.3 = 0.075 \tag{11.32}$$

*or* 7.5% *expected value.*

Since $V_t$ is a function of $\theta_t$ and $t$, we can determine the dependence on $\theta_t$ using Itô's lemma. The direct application of Itô's lemma (5.12) on $V(\theta, t)$ gives, in comparison to (11.28), the parameters in this equation

$$\begin{aligned} \mu_t V_t &= m\theta_t \frac{\partial V_t}{\partial \theta_t} + \frac{\partial V_t}{\partial t} + \frac{1}{2} s^2 \theta_t^2 \frac{\partial^2 V_t}{\partial \theta^2} \\ \sigma_t V_t &= s\theta_t \frac{\partial V_t}{\partial \theta}. \end{aligned}$$

Due to equation (11.30) we have $\mu_t V_t - \lambda_t \sigma_t V_t = r V_t$, so that we obtain the following differential equation for $V_t$:

$$\frac{\partial V_t}{\partial t} + (m - \lambda_t s)\theta_t \frac{\partial V_t}{\partial \theta} + \frac{1}{2} s^2 \theta_t^2 \frac{\partial^2 V_t}{\partial \theta^2} = r \cdot V_t \tag{11.33}$$

This equation (11.33) is very similar to the Black-Scholes differential equation and is in fact identical to (6.3) for $\theta_t = S_t$, where $S_t$ denotes the stock

price with no dividends. In this case $\theta_t$ itself is the price of the risk bearing instrument and must therefore satisfy (11.30), like the price $V_t$ of any derivative based on the stock. Thus we obtain

$$m - r = \lambda_t s, \tag{11.34}$$

so that the second term in (11.33) is equal to

$$r\theta_t \frac{\partial V_t}{\partial \theta}. \tag{11.35}$$

Thus we have a differential equation:

$$\frac{1}{2}s^2\theta_t^2 \frac{\partial^2 V_t}{\partial \theta^2} + r\theta_t \frac{\partial V_t}{\partial \theta} - rV_t + \frac{\partial V_t}{\partial t} = 0 \tag{11.36}$$

which is identical to (6.3) after renaming the variables. More explicitly, let $S_t = \theta_t, b = r$ (since there are no dividends) and let $\sigma = s$ using the notation in Section 6.1.

## 11.5 The Random Walk Hypothesis

We have seen that econometric models, at least with stock prices and exchange rates, motivate using a random walk as a statistical model. With exchange rates we saw that as a consequence of the uncovered interest rate parity and the assumption of risk neutrality of forward and future speculators the model in (11.14) follows a random walk. Assuming a geometric Brownian motion for stock price as in (11.18), it then follows from Itô's lemma that the log of stock price follows a Wiener process with a constant drift rate:

$$d \ln S_t = \mu^* dt + \sigma dW_t \tag{11.37}$$

where $\mu^* = \mu - \sigma^2/2$. If one observes (11.37) in time intervals of length $\Delta > 0$, i.e., at discrete points in time $0, \Delta, 2\Delta, ...$, then one obtains

$$\ln S_{t\Delta} = \ln S_{(t-1)\Delta} + \Delta\mu^* + \sqrt{\Delta}\sigma\xi_t \tag{11.38}$$

with independent, standard normally distributed $\xi_t, t = 1, 2, ....$ The process (11.38) is a random walk with a drift for the logged stock prices. The log returns (see Definition 11.15) over the time interval of length $\Delta$ are also independently normally distributed with expected value $\Delta\mu^*$ and variance $\Delta\sigma^2$.

With long interest rate time series the random walk appears to be less plausible, since it is assumed that in the long-run there is a stationary level around

which interest rates fluctuate in the short run. Let's consider once again the process for the short rate in (11.16), the Cox-Ingersoll-Ross (CIR) model. A discrete approximation is

$$r_t - r_{t-1} = \alpha + \beta r_{t-1} + \sigma\sqrt{r_{t-1}}\xi_t$$

or

$$r_t = \alpha + (1+\beta) r_{t-1} + \sigma\sqrt{r_{t-1}}\xi_t. \tag{11.39}$$

If $\beta$ in (11.39) is negative (and larger than -2), then the process is a stationary AR(1) process with heteroscedastic error terms. In Example 11.1 we encountered such a process with heteroscedastic error terms.

There is also the interpretation that interest rates are, at least in the short-term, explained well by a random walk. It is therefore of general interest to test whether a random walk exists. Below we show the distinguishing features of the three versions of the random walk hypothesis. In general we consider a random walk with a drift:

$$P_t = \mu + P_{t-1} + \varepsilon_t. \tag{11.40}$$

1. The stochastic errors in (11.40) are independent and identically distributed (i.i.d.) with expectation zero and variance $\sigma^2$. This hypothesis has already been tested on multiple data sets in the sixties and was empirically determined to be unsupported. For example, distinct volatility clusters were discovered which under the i.i.d. hypothesis are statistically not expected.
2. The stochastic errors in (11.40) are independent but not necessarily identically distributed with an expectation of zero. This hypothesis is weaker than the first one since, for example, it allows for heteroscedasticity. Nonetheless, even here the empirical discoveries were that a dependence between the error terms must be assumed.
3. The stochastic errors in (11.40) are uncorrelated, i.e., $\gamma_\tau(\varepsilon_t) = 0$ for every $\tau \neq 0$. This is the weakest and most often discussed random walk hypothesis. Empirically it is most often tested through the statistical (in)significance of the estimated autocorrelations of $\varepsilon_t$.

The discussion of the random walk hypotheses deals with, above all, the predicability of financial time series. Another discussion deals with the question of whether the model (11.40) with independent, or as the case may be, with uncorrelated, error terms is even a reasonable model for financial time series or whether it would be better to use just a model with a deterministic trend. Such a *trend-stationary* model has the form:

$$P_t = \nu + \mu t + \varepsilon_t \tag{11.41}$$

with constant parameters $\nu$ and $\mu$. The process (11.41) is non-stationary since, for example, $\mathsf{E}[P_t] = \nu + \mu t$, the expected value is time dependent. If the linear time trend is filtered from $P_t$, then the stationary process $P_t - \mu t$ is obtained.

To compare the difference stationary random walk with a drift to the trend stationary process (11.41) we write the random walk from (11.40) through recursive substitution as

$$P_t = P_0 + \mu t + \sum_{i=1}^{t} \varepsilon_i, \tag{11.42}$$

with a given initial value $P_0$. One sees that the random walk with a drift also implies a linear time trend, but the cumulative stochastic increments $(\sum_{i=1}^{t} \varepsilon_i)$ in (11.42) are not stationary, unlike the stationary increments $(\varepsilon_t)$ in (11.41). Due to the representation (11.42) the random walk with or without a drift will be described as *integrated*, since the deviation from a deterministic trend is the sum of error terms. Moreover, every error term $\varepsilon_t$ has a permanent influence on all future values of the process. For the best forecast in the sense of the mean squared error it holds for every $k > 0$,

$$\mathsf{E}[P_{t+k} \mid \mathcal{F}_t] = P_0 + \mu(t+k) + \sum_{i=1}^{t} \varepsilon_i.$$

In contrast, the impact of a shock $\varepsilon_t$ on the forecast of the trend-stationary process (11.41) could be zero, i.e.,

$$\mathsf{E}[P_{t+k} \mid \mathcal{F}_t] = \nu + \mu(t+k).$$

It is thus of at most importance to distinguish between a difference stationary and a trend-stationary process. It is worth emphasising here that the random walk is only a special case of a difference stationary process. If, for example, the increasing variables in (11.40) are stationary but are autocorrelated, then we have a general difference stationary process. There are many statistical tests which test whether a process is difference stationary or not. Two such tests are discussed in the next section.

## 11.6 Unit Root Tests

In Example 11.1 we discussed that the AR(1) process is:

$$X_t = c + \alpha X_{t-1} + \varepsilon_t. \tag{11.43}$$

Given $|\alpha| < 1$, the process is stationary when $E[X_0] = \frac{c}{1-\alpha}$ or after the "decaying process". The case where $\alpha = 1$ corresponds to the random walk which is non-stationary. The relationship between a stationary AR(1) process and $\alpha$ close to one is so similar to a random walk that it is often tested whether we have the case $\alpha = 1$ or $\alpha < 1$. To do this the so called *unit root tests* have been developed.

### 11.6.1 Dickey-Fuller Tests

The unit root test developed by Dickey and Fuller tests the null hypothesis of a *unit root*, that is, there is a root for the characteristic equation (12.6) of the AR(1) process with $z = 1$, against the alternative hypothesis that the process has no unit roots. As a basis for the test the following regression used is:

$$\Delta X_t = (\alpha - 1)X_{t-1} + \varepsilon_t, \tag{11.44}$$

which is obtained by rearranging (11.43) with $c = 0$. If $X_t$ is a random walk, then the coefficient of $X_{t-1}$ is equal to zero. If, on the other hand, $X_t$ is a stationary AR(1) process, then the coefficient is negative. The standard $t$-statistic is formed

$$\hat{t}_n = \frac{1 - \hat{\alpha}}{\sqrt{\hat{\sigma}^2(\sum_{t=2}^{n} X_{t-1}^2)^{-1}}}, \tag{11.45}$$

where $\hat{\alpha}$ and $\hat{\sigma}^2$ are the least squares estimators for $\alpha$ and the variance $\sigma^2$ of $\varepsilon_t$. For increasing $n$ the statistic (11.45) converges not to a standard normal distribution but instead to the distribution of a functional of Wiener process,

$$\hat{t}_n \xrightarrow{\mathcal{L}} \frac{W^2(1) - 1}{2\left\{\int_0^1 W^2(u)du\right\}^{1/2}},$$

where $W$ is a standard Wiener process. The critical value of the distribution are, for example, at the 1%, 5% and 10% significance levels, -2.58, -1.95, and -1.62 respectively.

A problem with this test is that the normal test significance level (for example 5%) is not reliable when the error terms $\varepsilon_t$ in (11.44) are autocorrelated. The larger the autocorrelation of $\varepsilon_t$, the larger the distortion in general will be of the test significance. Ignoring then that autocorrelations could lead to the rejection of the null hypothesis of a unit root at low significance levels of 5%, when in reality the significance level lies at, for example, 30%. In order to prohibit these negative effects, Dickey and Fuller suggest another regression

| | | $\beta$ | | | |
|---|---|---|---|---|---|
| $\alpha$ | $p$ | -0.99 | -0.9 | 0 | 0.9 |
| 1 | 3 | 0.995 | 0.722 | 0.045 | 0.034 |
| | 11 | 0.365 | 0.095 | 0.041 | 0.039 |
| 0.9 | 3 | 1.000 | 0.996 | 0.227 | 0.121 |
| | 11 | 0.667 | 0.377 | 0.105 | 0.086 |

Table 11.3: ADF-Test: Simulated rejection probabilities for the process (11.48) at a nominal significance level of 5% (according to Friedmann (1992)).

which contains lagged differences. The regression of this *augmented Dickey Fuller Test* (ADF) is thus:

$$\Delta X_t = c + (\alpha - 1)X_{t-1} + \sum_{i=1}^{p} \alpha_i \Delta X_{t-i} + \varepsilon_t \tag{11.46}$$

where as with the simple Dickey-Fuller Test the null hypothesis of a unit root is rejected when the test statistic (11.45) is smaller than the critical value (which are summarised in table 11.3). Naturally the choice of $p$ is problematic. In general it holds that the size of the test is better when $p$ gets larger, but which causes the test to lose *power*. This is illustrated in a simulated process. The errors $\varepsilon_t$ are correlated through the relationship:

$$\varepsilon_t = \beta\xi_{t-1} + \xi_t$$

where $\xi_t$ are i.i.d. $(0, \sigma^2)$. In the next chapter these processes will be referred to as *moving average* processes of order 1, MA(1). It holds that $\mathrm{Var}(\varepsilon_t) = \sigma^2(1+\beta^2)$, $\gamma_1(\varepsilon_t) = \mathrm{Cov}(\varepsilon_t, \varepsilon_{t-1}) = \beta\sigma^2$, and $\gamma_\tau(\varepsilon_t) = 0$ for $\tau \geq 2$. For the ACF of $\varepsilon_t$ we then get

$$\rho_\tau(\varepsilon_t) = \begin{cases} \frac{\beta}{1+\beta^2} & \text{wenn } \tau = 1 \\ 0 & \text{wenn } \tau \geq 2. \end{cases} \tag{11.47}$$

For the process

$$X_t = \alpha X_{t-1} + \beta\xi_{t-1} + \xi_t \tag{11.48}$$

simulations of the ADF Tests were done and are summarised in an abbreviated form in Table 11.3.

As one can see, the nominal significance level of 5% under the null hypothesis ($\alpha = 1$) is held better, if $p$ is larger. However the power of the test decreases,

i.e., the test is no longer capable of distinguishing between a process with unit roots and a stationary process with $\alpha = 0.9$. Thus in choosing $p$ there is also the conflict between validity and power of the test.

If $X_t$ is a trend-stationary process as in (11.41), the ADF test likewise does not often enough reject the (false) null hypothesis of a unit root. Asymptotically the probability of rejecting goes to zero. The ADF regression (11.46) can be extended by a linear time trend, i.e., run the regression

$$\Delta X_t = c + \mu t + (\alpha - 1)X_{t-1} + \sum_{i=1}^{p} \alpha_i \Delta X_{t-i} + \varepsilon_t \tag{11.49}$$

and test the significance of $\alpha$. The critical values are contained in tables. The ADF test with a time trend (11.49) has power against a trend-stationary process. On the other hand, it loses power as compared to the simple ADF test (11.46), when the true process, for example, is a stationary AR(1) process.

As an empirical example, consider the daily stock prices of the 20 largest German stock companies from Jan. 2, 1974 to Dec. 30, 1996. Table 11.4 displays the ADF test statistics for the logged stock prices for $p = 0$ and $p = 4$. The tests were run with and without a linear time trend. In every regression a constant was included in estimation.

Only for RWE with a linear time trend does the ADF test reject the null hypothesis of a unit root by a significance level of 10%. Since in all other cases no unit root is rejected, it appears that taking differences of stock prices is a necessary operation in order to obtain a stationary process, i.e., to get log returns that can be investigated further. These results will be put into question in the next section using another test.

### 11.6.2 The KPSS Test of Stationarity

The KPSS Test from Kwiatkowski, Phillips, Schmidt and Shin (1992) tests for stationarity, i.e., for a unit root. The hypotheses are thus exchanged from those of the ADF test. As with the ADF test, there are two cases to distinguish between, whether to estimate with or without a linear time trend. The regression model with a time trend has the form

$$X_t = c + \mu t + k \sum_{i=1}^{t} \xi_i + \eta_t, \tag{11.50}$$

with stationary $\eta_t$ and i.i.d. $\xi_t$ with an expected value 0 and variance 1. Obviously for $k \neq 0$ the process is integrated and for $k = 0$ trend-stationary.

| | ADF | | | | KPSS | | | |
|---|---|---|---|---|---|---|---|---|
| | without time trend | | with time trend | | without time trend | | with time trend | |
| $p$ and $T$ | 0 | 4 | 0 | 4 | 8 | 12 | 8 | 12 |
| ALLIANZ | -0.68 | -0.62 | 2.44 | 2.59 | 24.52** | 16.62** | 2.36** | 1.61** |
| BASF | 0.14 | 0.34 | 2.94 | 3.13 | 23.71** | 16.09** | 1.39** | 0.95** |
| BAYER | -0.11 | 0.08 | 2.96 | 3.26 | 24.04** | 16.30** | 1.46** | 1.00** |
| BMW | -0.71 | -0.66 | 2.74 | 2.72 | 23.92** | 16.22** | 2.01** | 1.37** |
| COMMERZ- | . | . | . | . | | | | |
| BANK | -0.80 | -0.67 | 1.76 | 1.76 | 22.04** | 14.96** | 1.43** | 0.98** |
| DAIMLER | -1.37 | -1.29 | 2.12 | 2.13 | 22.03** | 14.94** | 3.34** | 2.27** |
| DEUTSCHE | . | . | . | . | | | | |
| BANK | -1.39 | -1.27 | 2.05 | 1.91 | 23.62** | 16.01** | 1.70** | 1.16** |
| DEGUSSA | -0.45 | -0.36 | 1.94 | 1.88 | 23.11** | 15.68** | 1.79** | 1.22** |
| DRESDNER | -0.98 | -0.94 | 1.90 | 1.77 | 22.40** | 15.20** | 1.79** | 1.22** |
| HOECHST | 0.36 | 0.50 | 3.24 | 3.37 | 23.80** | 16.15** | 1.42** | 0.97** |
| KARSTADT | -1.18 | -1.17 | 1.15 | 1.15 | 20.40** | 13.84** | 3.33** | 2.26** |
| LINDE | -1.69 | -1.44 | 2.74 | 2.70 | 24.40** | 16.54** | 3.14** | 2.15** |
| MAN | -1.78 | -1.58 | 1.66 | 1.61 | 21.97** | 14.91** | 1.59** | 1.08** |
| MANNES- | . | . | . | . | | | | |
| MANN | -0.91 | -0.80 | 2.73 | 2.55 | 21.97** | 14.93** | 1.89** | 1.29** |
| PREUSSAG | -1.40 | -1.38 | 2.21 | 2.03 | 23.18** | 15.72** | 1.53** | 1.04** |
| RWE | -0.09 | -0.04 | 2.95 | 2.84 | 24.37** | 16.52** | 1.66** | 1.14** |
| SCHERING | 0.11 | 0.04 | 2.37 | 2.12 | 24.20** | 16.40** | 2.35** | 1.60** |
| SIEMENS | -1.35 | -1.20 | 2.13 | 1.84 | 23.24** | 15.76** | 1.69** | 1.15** |
| THYSSEN | -1.45 | -1.34 | 1.92 | 1.90 | 21.97** | 14.90** | 1.98** | 1.35** |
| VOLKS- | . | . | . | . | | | | |
| WAGEN | -0.94 | -0.81 | 1.89 | 1.73 | 21.95** | 14.89** | 1.11** | 0.76** |

Table 11.4: Unit root tests: ADF Test (Null hypothesis: unit root) and KPSS Test (Null hypothesis: stationary). The augmented portion of the ADF regression as order $p = 0$ and $p = 4$. The KPSS statistic was calculated with the reference point $T = 8$ and $T = 12$. The asterisks indicate significance at the 10% (*) and 1% (**) levels.

SFEAdfKpss

The null hypothesis is $H_0 : k = 0$, and the alternative hypothesis is $H_1 : k \neq 0$.

Under $H_0$ the regression (11.50) is run with the method of the least squares obtaining the residuals $\hat{\eta}_t$. Using these residuals the partial sum

$$S_t = \sum_{i=1}^{t} \hat{\eta}_i,$$

is built which under $H_0$ is integrated of order 1, i.e., the variance $S_t$ increases linearly with $t$. The KPSS test statistic is then

$$KPSS_T = \frac{\sum_{t=1}^{n} S_t^2}{n^2 \hat{\omega}_T^2}, \tag{11.51}$$

where

$$\hat{\omega}_T^2 = \hat{\sigma}_\eta^2 + 2\sum_{\tau=1}^{T}(1 - \frac{\tau}{T-1})\hat{\gamma}_\tau$$

is an estimator of the spectral density at a frequency of zero where $\hat{\sigma}_\eta^2$ is the variance estimator of $\eta_t$ and $\hat{\gamma}_\tau = 1/n \sum_{t=\tau+1}^{n} \hat{\eta}_t \hat{\eta}_{t-\tau}$ is the covariance estimator. The problem again is to determine the reference point $T$: for $T$ that are too small the test is biased when there is autocorrelation, for $T$ that is too large it loses power.

The results of the KPSS tests in Table 11.4 clearly indicate that the investigated stock prices are not stationary or trend-stationary, since in every case the null hypothesis at a significance level of 1% was rejected. Even RWE, which was significant under the ADF test at a significance level of 10 %, implies a preference of the hypothesis of unit roots here at a lower significance level.

### 11.6.3 Variance Ratio Tests

If one wants to test whether a time series follows a random walk, one can take advantage of the fact that the variance of a random walk increases linearly with time, see (11.4). Considering the log prices of a financial time series, $\ln S_t$, the null hypothesis would be

$$H_0 : r_t = \mu + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{N}(0, \sigma^2)$$

with log returns $r_t = \ln S_t - \ln S_{t-1}$, constant $\mu$ and $\varepsilon_t$ white noise. An alternative hypothesis is, for example, that $r_t$ is stationary and autocorrelated. The sum over the returns is formed

$$r_t(q) = r_t + r_{t-1} + \ldots + r_{t-q+1}$$

and the variance of $r_t(q)$ is determined. For $q = 2$ it holds that, for example,

$$\begin{aligned} \mathrm{Var}\{r_t(2)\} &= \mathrm{Var}(r_t) + \mathrm{Var}(r_{t-1}) + 2\,\mathrm{Cov}(r_t, r_{t-1}) \\ &= 2\,\mathrm{Var}(r_t) + 2\gamma_1 \\ &= 2\,\mathrm{Var}(r_t)(1 + \rho_1), \end{aligned}$$

where taking advantage of the stationarity of $r_t$, generally

$$\mathrm{Var}\{r_t(q)\} = q\,\mathrm{Var}(r_t)\left(1 + 2\sum_{\tau=1}^{q-1}(1 - \frac{\tau}{q})\rho_\tau\right). \qquad (11.52)$$

Under $H_0$ it holds that $\rho_\tau = 0$ for all $\tau > 0$, so that under $H_0$

$$\frac{\mathrm{Var}\{r_t(q)\}}{q\,\mathrm{Var}(r_t)} = 1.$$

A test statistic can now be constructed where the consistent estimator

$$\hat{\mu} = \frac{1}{n}(\ln S_n - \ln S_0)$$

for $\mu$,

$$\hat{\gamma_0} = \frac{1}{n-1}\sum_{t=2}^{n}(\ln S_t - \ln S_{t-1} - \hat{\mu})^2$$

for $\mathrm{Var}(r_t)$ and

$$\hat{\gamma_0}(q) = \frac{n}{q(n-q)(n-q+1)}\sum_{t=q+1}^{n}(\ln S_t - \ln S_{t-q} - q\hat{\mu})^2$$

for $\frac{1}{q}\mathrm{Var}\{r_t(q)\}$ are substituted into (11.52). The test statistic is then

$$VQ(q) = \frac{\hat{\gamma_0}(q)}{\hat{\gamma_0}} - 1.$$

It can be shown that the asymptotic distribution is

$$\sqrt{n}VQ(q) \xrightarrow{\mathcal{L}} \mathrm{N}\left(0, \frac{2(2q-1)(q-1)}{3q}\right).$$

The asymptotic variance can be established through the following approximation: Assume that $\hat{\mu} = 0$ and $n \gg q$. Then we have that $\ln S_t - \ln S_{t-q} = \sum_{j=0}^{q-1} r_{t-j}$ and

$$\begin{aligned}
VQ(q) &\approx \frac{1}{qn}\sum_{t=q+1}^{n}\left((\sum_{j=0}^{q-1} r_{t-j})^2 - q\hat{\gamma_0}\right)/\hat{\gamma_0} \\
&= \frac{1}{qn}\sum_{t=q+1}^{n}\frac{1}{\hat{\gamma_0}}\left(\sum_{j=0}^{q-1} r_{t-j}^2 + 2\sum_{j=0}^{q-2} r_{t-j}r_{t-j-1} + \ldots + 2r_t r_{t-q+1} - q\hat{\gamma_0}\right) \\
&\approx \frac{1}{q}\left(q\hat{\gamma_0} + 2(q-1)\hat{\gamma_1} + \ldots + 2\hat{\gamma_{q-1}} - q\hat{\gamma_0}\right)/\hat{\gamma_0} \\
&= 2\sum_{j=1}^{q-1}\frac{q-j}{q}\hat{\rho_j}.
\end{aligned}$$

Since under $H_0$ the estimated autocorrelation $\hat{\rho}_j$ scaled with $\sqrt{n}$ is asymptotically standard normal and is independent, see Section 12.5, the asymptotic variance is thus:

$$\begin{aligned}
\mathrm{Var}_{as}\{\sqrt{n}VQ(q)\} &= \mathrm{Var}_{as}(2\sum_{j=1}^{q-1}\frac{q-j}{q}\sqrt{n}\hat{\rho}_j) \\
&= 4\sum_{j=1}^{q-1}\frac{(q-j)^2}{q^2}\mathrm{Var}_{as}(\sqrt{n}\hat{\rho}_j) \\
&= 4\sum_{j=1}^{q-1}\frac{(q-j)^2}{q^2} \\
&= 4(q-1) - \frac{8}{q}\sum_{j=1}^{q-1} j + \frac{4}{q^2}\sum_{j=1}^{q-1} j^2.
\end{aligned}$$

With the summation formulas

$$\sum_{j=1}^{q-1} j = (q-1)q/2$$

and

$$\sum_{j=1}^{q-1} j^2 = q(q-1)(2q-1)/6$$

we finally obtain

$$\mathrm{Var}_{as}\{\sqrt{n}VQ(q)\} = \frac{2(2q-1)(q-1)}{3q}.$$

## 11.7 Recommended Literature

Four current Textbooks in the area of empirical financial market analysis are Mills (1993), Gouriéroux (1997), Campbell, Lo and MacKinlay (1997) and Gouriéroux and Jasiak (2002). The focus of Gouriéroux and Mills is more towards the econometric/time series analysis applications (which will also be followed in this book), whereas Campbell, Lo and MacKinlay discuss many economic applications that do not always end with statistical or econometric models. As an introduction and yet a comprehensive book on time series analysis Schlittgen and Streitberg (1995) is recommended. The same can be found with Copeland and Weston (1992), an introductory book on finance theory.

The experiment of the expectation hypotheses comes from Forsythe, Palfrey and Plott (1982). The definition of expectations and efficient markets is based on Jarrow (1992). The CAPM is developed in Sharpe (1964), Lintner (1965) and Mossin (1966). The discussion on the interest rate parity follows Jarchow and Rühmann (1994)[pp.236] and that of the term structure models of Cox-Ingersoll-Ross follow Ingersoll (1987). The standard option pricing model originated in Black and Scholes (1973). The market price of risk is discussed in Hull (2000),

A good overview of unit root tests is given in Hassler (1994). The ADF test is taken from Dickey and Fuller (1979).

# 12 ARIMA Time Series Models

In this chapter we will deal with classic, linear time series analysis. At first we will define the general linear process.

**Definition 12.1 (Linear Process)**
*If the process $X_t$ has the representation*

$$X_t = \mu + \sum_{i=-\infty}^{\infty} a_i \varepsilon_{t-i}$$

*with white noise $\varepsilon_t$ and absolute summability of the filter $(a_i) : \sum_{i=-\infty}^{\infty} |a_i| < \infty$, then it is a linear process.*

The linear process $X_t$ is covariance stationary, since $\mathsf{E}(X_t) = \mu$ and

$$\mathrm{Cov}(X_t, X_{t+\tau}) = \sigma^2 \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_i a_j \mathbf{1}(\tau = i - j) = \sigma^2 \sum_{i=-\infty}^{\infty} a_i a_{i-\tau}$$

with $\mathrm{Var}(\varepsilon_t) = \sigma^2$.

In general in econometrics, especially in the area of financial markets, series are observed which indicate non-stationary behaviour. In the previous chapter we saw that econometric models, which are based on assumptions of rational expectations, frequently imply that the relevant levels of, for example, prices, follow a random walk. In order to handle these processes within the framework of the classic time series analysis, we must first form the differences in order to get a stationary process. We generalise the definition of a difference stationary process in the following definition.

**Definition 12.2 (Integrated process)**
*We say that the process $X_t$ is integrated of order d, $I(d)$, when $(1-L)^{d-1}X_t$ is non-stationary and $(1-L)^d X_t$ is stationary.*

White noise is, for example, $I(0)$, a random walk $I(1)$. In only a few cases processes are observed that are $I(d)$ with $d > 1$. This means that in most cases

first differences are enough to form a stationary process. In the following we assume that the observed process $Y_t$ is $I(d)$ and we consider the transformed process $X_t = (1-L)^d Y_t$, i.e., we will concentrate on stationary processes.

## 12.1 Moving Average Processes

The moving average process of order $q$, MA($q$), is defined as

$$X_t = \beta_1 \varepsilon_{t-1} + \ldots + \beta_q \varepsilon_{t-q} + \varepsilon_t \tag{12.1}$$

with white noise $\varepsilon_t$. With the Lag-Operator $L$ (see Definition 11.13) instead of (12.1) we can write

$$X_t = \beta(L)\varepsilon_t \tag{12.2}$$

with $\beta(L) = 1 + \beta_1 L + \ldots + \beta_q L^q$. The MA($q$) process is stationary, since it is formed as the linear combination of a stationary process. The mean function is simply $\mathsf{E}(X_t) = 0$. Let $\beta_0 = 1$, then the covariance structure is

$$\begin{aligned}
\gamma_\tau &= \operatorname{Cov}(X_t, X_{t+\tau}) \\
&= \operatorname{Cov}(\sum_{i=0}^{q} \beta_i \varepsilon_{t-i}, \sum_{j=0}^{q} \beta_j \varepsilon_{t+\tau-j}) \\
&= \sum_{i=0}^{q} \sum_{j=0}^{q} \beta_i \beta_j \operatorname{Cov}(\varepsilon_{t-i}, \varepsilon_{t+\tau-j}) \\
&= \sum_{i=0}^{q-|\tau|} \beta_i \beta_{i+|\tau|} \sigma^2, \;\; |\tau| \leq q.
\end{aligned}$$

For the ACF we have for $|\tau| \leq q$

$$\rho_\tau = \frac{\sum_{i=0}^{q-|\tau|} \beta_i \beta_{i+|\tau|}}{\sum_{i=0}^{q} \beta_i^2}, \tag{12.3}$$

and $\rho_\tau = 0$ for $|\tau| > q$, i.e., the ACF breaks off after $q$ lags.

As an example consider the MA(1) process

$$X_t = \beta \varepsilon_{t-1} + \varepsilon_t,$$

which according to (12.3) holds that $\rho_1 = \beta/(1+\beta^2)$ and $\rho_\tau = 0$ for $\tau > 1$. Figure 12.1 shows the correleogram of a MA(1) process.

Obviously the process

$$X_t = 1/\beta \varepsilon_{t-1} + \varepsilon_t$$

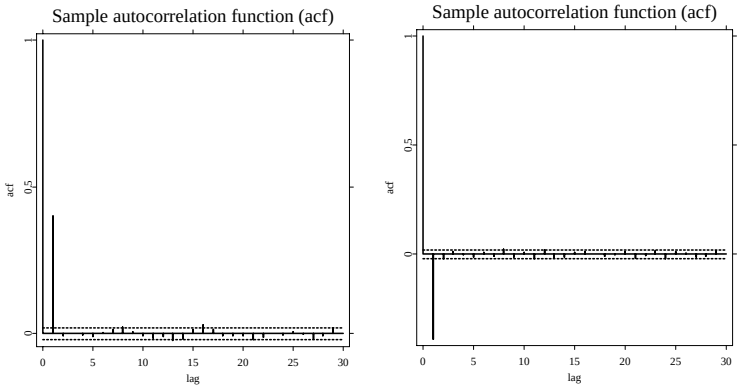


Figure 12.1: ACF of a MA(1) process with $\beta = 0.5$ (left) and $\beta = -0.5$ (right). SFEacfma1

has the same ACF, and it holds that

$$\rho_1 = \frac{1/\beta}{1 + (1/\beta)^2} = \frac{\beta}{1 + \beta^2}.$$

In other words the process with the parameter $\beta$ has the same stochastic properties as the process with the parameter $1/\beta$. This identification problem can be countered by requiring that the solutions of the characteristic equation

$$1 + \beta_1 z + \ldots + \beta_q z^q = 0 \tag{12.4}$$

lie outside of the complex unit circle. In this case the linear filter $\beta(L)$ is invertible, i.e., a polynomial $\beta^{-1}(L)$ exists so that $\beta(L)\beta^{-1}(L) = 1$ and $\beta^{-1}(L) = b_0 + b_1 L + b_2 L^2 + \ldots$. Figure 12.2 displays the correlogram of a MA(2) process $X_t = \beta_1 \varepsilon_{t-1} + \beta_2 \varepsilon_{t-2} + \varepsilon_t$ for some collections of parameters.

## 12.2 Autoregressive Process

The linear autoregressive process of order $p$, (AR($p$)), is defined as

$$X_t = \nu + \alpha_1 X_{t-1} + \ldots + \alpha_p X_{t-p} + \varepsilon_t \tag{12.5}$$

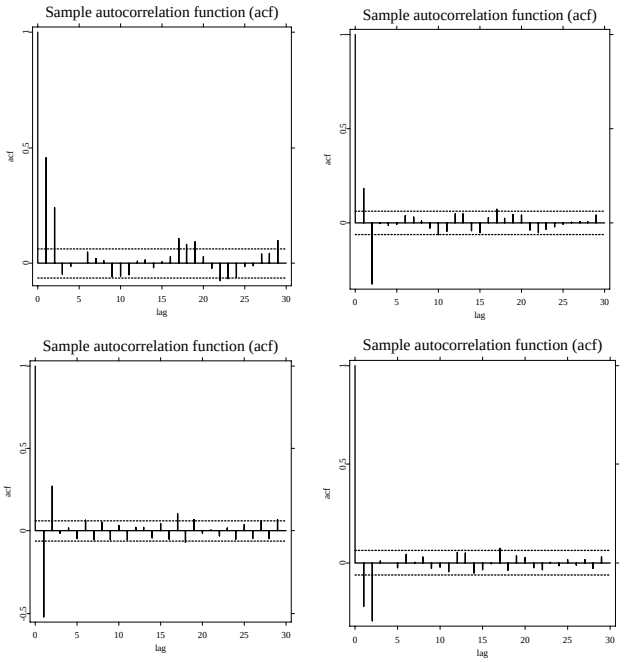


Figure 12.2: ACF of a MA(2) process with $(\beta_1 = 0.5, \beta_2 = 0.4)$ (top left), $(\beta_1 = 0.5, \beta_2 = -0.4)$ (top right), $(\beta_1 = -0.5, \beta_2 = 0.4)$ (bottom left) and $(\beta_1 = -0.5, \beta_2 = -0.4)$ (bottom right). SFEacfma2

Using the definition of the lag-operator $L$ (see Definition 11.13), (12.5) can also be written as

$$\alpha(L)X_t = \nu + \varepsilon_t,$$

with the lag-polynomial $\alpha(L) = 1 - \alpha_1 L - \ldots - \alpha_p L^p$. The process $X_t$ is stationary if all roots of the characteristic equation

$$\alpha(z) = 1 - \alpha_1 z - \ldots - \alpha_p z^p = 0. \tag{12.6}$$

lie outside of the complex unit circle, that is, if for all $z$ with $|z| \leq 1$ it holds that

$$\alpha(z) \neq 0. \tag{12.7}$$

In this case there is an inverted filter $\alpha^{-1}(L)$ for the linear filter $\alpha(L)$, such that the following holds,

$$\alpha(L)\alpha^{-1}(L) = 1$$

and

$$\alpha^{-1}(L) = a_0 + a_1 L + a_2 L^2 + \ldots = \sum_{i=0}^{\infty} a_i L^i.$$

The process (12.5) can also be written under the condition (12.7)

$$\begin{aligned} X_t &= \alpha^{-1}(1)\nu + \alpha^{-1}(L)\varepsilon_t \\ &= \sum_{i=0}^{\infty} a_i \nu + \sum_{i=0}^{\infty} a_i L^i \varepsilon_t, \end{aligned}$$

as a MA($\infty$) process.

A simple way to find and invert the autocovariance of an AR($p$) process with given parameters is to use the Yule-Walker equations. They are derived from the definition of an AR($p$) process in (12.5) by multiplying by $X_{t-\tau}$ and taking expectations.

$$\mathsf{E}[X_t X_{t-\tau}] = \alpha_1 \mathsf{E}[X_{t-1} X_{t-\tau}] + \ldots + \alpha_p \mathsf{E}[X_{t-p} X_{t-\tau}]. \tag{12.8}$$

Since $\mathsf{E}[X_t X_{t-\tau}]$ is the definition of the autocovariance function $\gamma_\tau$ for $\nu = 0$, it can even be written simpler for $\tau = 1, \ldots, p$

$$\begin{array}{rcl} \alpha_1\gamma_0 + \alpha_2\gamma_1 + \cdots + \alpha_p\gamma_{p-1} &=& \gamma_1 \\ \alpha_1\gamma_1 + \alpha_2\gamma_0 + \cdots + \alpha_p\gamma_{p-2} &=& \gamma_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & \\ \alpha_1\gamma_{p-1} + \alpha_2\gamma_{p-2} + \cdots + \alpha_p\gamma_0 &=& \gamma_p \end{array} \tag{12.9}$$

or by dividing by the variance $\gamma_0$

$$\rho = R\alpha \tag{12.10}$$

with $\rho = (\rho_1 \rho_2 \ldots \rho_p)^\top$, $\alpha = (\alpha_1 \alpha_2 \ldots \alpha_p)^\top$, and the $p \times p$-autocovariance matrix

$$R = \begin{pmatrix} 1 & \rho_1 & \cdots & \rho_{p-1} \\ \rho_1 & 1 & \cdots & \rho_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p-1} & \rho_{p-2} & \cdots & 1 \end{pmatrix}.$$

Yule-Walker equations are useful in determining the ACF for given parameters or, vice versa, in determining the estimated parameters for the given (empirical) autocorrelation.

**Example 12.1 (AR(1))**
*The AR(1) process from Example 11.1 with $\nu = 0$ has the characteristic*

*equation* $1-\alpha z = 0$. *The explicit solution* $z = 1/\alpha$ *and* $|z| > 1$ *occurs exactly when* $|\alpha| < 1$. *The inverse filter of* $\alpha(L) = 1-\alpha L$ *is thus* $\alpha^{-1}(L) = \sum_{i=0}^{\infty} \alpha^i L^i$ *and the MA*$(\infty)$ *representation of the AR(1) process is*

$$X_t = \sum_{i=0}^{\infty} \alpha^i \varepsilon_{t-i}.$$

*The ACF of the AR(1) process is* $\rho_\tau = \alpha^\tau$. *For* $\alpha > 0$ *all autocorrelations are positive, for* $\alpha < 0$ *they alternate between positive and negative, see Figure 12.3.*

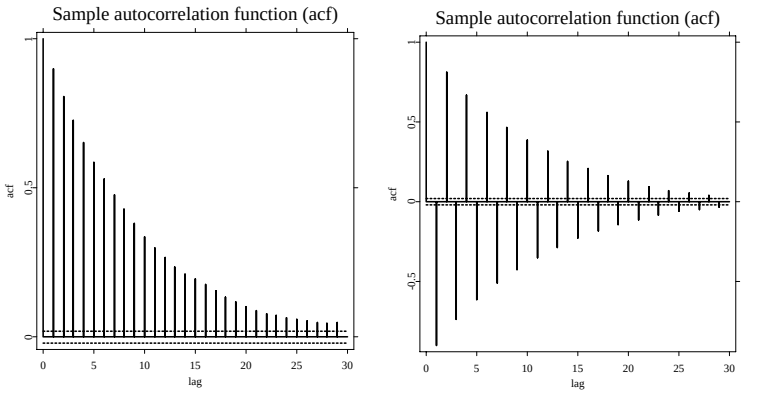


Figure 12.3: ACF of an AR(1) process with $\alpha = 0.9$ (left) and $\alpha = -0.9$ (right). SFEacfar1

**Example 12.2 (AR(2))**
*The AR(2) process with* $\nu = 0$,

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \varepsilon_t$$

*is stationary when given the roots* $z_1$ *and* $z_2$ *of the quadratic equation*

$$1 - \alpha_1 z - \alpha_2 z^2 = 0,$$

*it holds that* $|z_1| > 1$ *and* $|z_2| > 1$. *We obtain solutions as*

$$z_{1,2} = -\frac{\alpha_1}{2\alpha_2} \pm \sqrt{\frac{\alpha_1^2}{4\alpha_2^2} + \frac{1}{\alpha_2}}$$

*and* $z_1 z_2 = -1/\alpha_2$. *Due to* $|z_1| > 1$ *and* $|z_2| > 1$ *it holds that* $|z_1 z_2| = 1/|\alpha_2| > 1$ *and*

$$|\alpha_2| < 1. \tag{12.11}$$

*From the Yule-Walker equations in the case of an AR(2) process*

$$\rho_1 = \alpha_1 + \alpha_2 \rho_1 \tag{12.12}$$
$$\rho_2 = \alpha_1 \rho_1 + \alpha_2 \tag{12.13}$$

*it follows that* $\rho_1 = \alpha_1/(1-\alpha_2)$. *The case* $\rho_1 = \pm 1$ *is excluded because a root would lie on the unit circle (at 1 or -1). Thus for a stationary AR(2) process it must hold that*

$$|\rho_1| = |\alpha_1/(1-\alpha_2)| < 1,$$

*from which, together with (12.11), we obtain the 'stationarity triangle'*

$$\alpha_1 + \alpha_2 < 1 \tag{12.14}$$
$$\alpha_2 - \alpha_1 < 1 \tag{12.15}$$

*i.e., the region in which the AR(2) process is stationary.*

*The ACF of the AR(2) process is recursively given with (12.12), (12.13) and* $\rho_\tau = \alpha_1 \rho_{\tau-1} + \alpha_2 \rho_{\tau-2}$ *for* $\tau > 2$. *Figure (12.4) displays typical patterns.*

## 12.3 ARMA Models

The ARMA$(p, q)$ model is defined as

$$X_t = \nu + \alpha_1 X_{t-1} + \ldots + \alpha_p X_{t-p} + \beta_1 \varepsilon_{t-1} + \ldots + \beta_q \varepsilon_{t-q} + \varepsilon_t, \tag{12.16}$$

or as

$$\alpha(L) X_t = \nu + \beta(L) \varepsilon_t$$

with the moving average lag-polynomial $\beta(L) = 1 + \beta_1 L + \ldots + \beta_q L^q$ and the autoregressive lag-polynomial $\alpha(L) = 1 - \alpha_1 L - \ldots - \alpha_p L^p$. In order that the process (12.16) can have explicit parameterisation, it is required that the characteristic polynomials $\alpha(z)$ and $\beta(z)$ do not have any common roots. The process (12.16) is stationary when all the roots of the characteristic equation (12.6) lie outside of the unit circle. In this case (12.16) has the MA$(\infty)$ representation

$$X_t = \alpha^{-1}(L) \beta(L) \varepsilon_t.$$

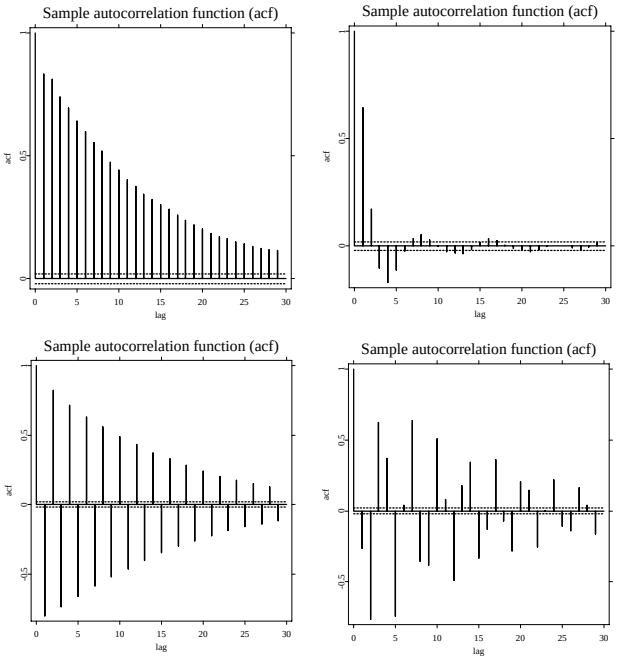


Figure 12.4: ACF of a AR(2) process with $(\alpha_1 = 0.5, \alpha_2 = 0.4)$ (top left), $(\alpha_1 = 0.9, \alpha_2 = -0.4)$ (top right), $(\alpha_1 = -0.4, \alpha_2 = 0.5)$ (bottom left) and $(\alpha_1 = -0.5, \alpha_2 = -0.9)$ (bottom right). SFEacfar2

The process $X_t$ in (12.16) is invertible when all the roots of the characteristic equation (12.4) lie outside of the unit circle. In this case (12.16) can be written as

$$\beta^{-1}(L)\alpha(L)X_t = \varepsilon_t,$$

that is an AR($\infty$) process. Thus we can approximate every stationary, invertible ARMA$(p, q)$ process with a pure AR or MA process of sufficiently large order. Going in the other direction, an ARMA$(p, q)$ process offers the possibility of parsimonious parameterisation.

## 12.4 Partial Autocorrelation

For a given stochastic process one is often interested in the connection between two random variables of a process at different points in time. One way to measure a linear relationship is with the ACF, i.e., the correlation between these two variables. Another way to measure the connection between $X_t$ and $X_{t+\tau}$ is to filter out of $X_t$ and $X_{t+\tau}$ the linear influence of the random variables that lie in between, $X_{t+1}, \ldots, X_{t+\tau-1}$, and then calculate the correlation of the transformed random variables. This is called the *partial* autocorrelation.

**Definition 12.3 (Partial autocorrelation)**
*The partial autocorrelation of $k-th$ order is defined as*

$$\begin{aligned} \phi_{kk} &= Corr(X_t - \mathcal{P}(X_t \mid X_{t+1}, \ldots, X_{t+k-1}), \\ &\qquad X_{t+k} - \mathcal{P}(X_{t+k} \mid X_{t+1}, \ldots, X_{t+k-1})) \end{aligned} \tag{12.17}$$

*where $\mathcal{P}(W \mid Z)$ is the best linear projection of $W$ on $Z$, i.e., $\mathcal{P}(W \mid Z) = \Sigma_{WZ}\Sigma_{ZZ}^{-1}Z$ with $\Sigma_{ZZ} = Var(Z)$ as the covariance matrix of the regressors and $\Sigma_{WZ} = Cov(W, Z)$ as the matrix of covariances between $W$ and $Z$.*

The 'best linear projection' is understood in the sense of minimising the mean squared error.

An equivalent definition is the solution to the system of equations

$$P_k \phi_k = \rho_{(k)}$$

with

$$P_k = \begin{pmatrix} 1 & \rho_1 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \cdots & \rho_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & 1 \end{pmatrix}$$

$\phi_k = (\phi_{k1}, \ldots, \phi_{kk})^\top$ and $\rho_{(k)} = (\rho_1, \ldots, \rho_k)^\top$. These are the Yule-Walker equations for an AR($k$) process. The last coefficient, $\phi_{kk}$, is the partial autocorrelation of order $k$, as defined above. Since only this coefficient is of interest in this context, the system of equations can be solved for $\phi_{kk}$ using the Cramer-Rule. We get

$$\phi_{kk} = \frac{|P_k^*|}{|P_k|}$$

where $P_k^*$ is equal to the matrix $P_k$, in which the $k-$th column is replaced with $\rho_{(k)}$. Here $|.|$ indicates the determinant. Since this can be applied to various

orders $k$, in the end we obtain a *partial autocorrelation function* (PACF). The PACF can be graphically displayed for a given stochastic process, similar to the ACF as a function of order $k$. This is called the partial autocorrelogram.

From the definition of PACF it immediately follows that there is no difference between PACF and ACF of order 1:

$$\phi_{11} = \rho_1.$$

For order 2 we have

$$\phi_{22} = \frac{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix}} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} \tag{12.18}$$

**Example 12.3 (AR(1))**
*The AR(1) process $X_t = \alpha X_{t-1} + \varepsilon_t$ has the ACF $\rho_\tau = \alpha^\tau$. For the PACF we have $\phi_{11} = \rho_1 = \alpha$ and*

$$\phi_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = \frac{\alpha^2 - \alpha^2}{1 - \alpha^2} = 0,$$

*and $\phi_{kk} = 0$ for all $k > 1$. This is plausible since the last coefficient of an AR(k) model for this process is zero for all $k > 1$. For $k = 2$ we illustrate the equivalence with Definition 12.3: From $X_t = \alpha X_{t-1} + \varepsilon_t$ we directly obtain $\mathcal{P}(X_{t+2}|X_{t+1}) = \alpha X_{t+1}$ with*

$$\alpha = \frac{Cov(X_{t+2}, X_{t+1})}{Var(X_{t+1})}.$$

*From the 'backward regression' $X_t = \alpha' X_{t+1} + \eta_t$ with white noise $\eta_t$ it further follows that $\mathcal{P}(X_t|X_{t+1}) = \alpha' X_{t+1}$ with*

$$\alpha' = \frac{Cov(X_t, X_{t+1})}{Var(X_{t+1})}.$$

*For $|\alpha| < 1$ the process is covariance-stationary and it holds that $Cov(X_{t+2}, X_{t+1}) = Cov(X_t, X_{t+1}) = \gamma_1$ and $\alpha = \alpha' = \rho_1$. We obtain*

$$\begin{aligned} & Cov\{X_t - \mathcal{P}(X_t|X_{t+1}), X_{t+2} - \mathcal{P}(X_{t+2}|X_{t+1})\} \\ = \; & Cov(X_t - \rho_1 X_{t+1}, X_{t+2} - \rho_1 X_{t+1}) \\ = \; & \mathsf{E}[(X_t - \rho_1 X_{t+1})(X_{t+2} - \rho_1 X_{t+1})] \\ = \; & \gamma_2 - 2\rho_1\gamma_1 + \rho_1^2\gamma_0 \end{aligned}$$

*and*

$$\begin{aligned} Var\{X_{t+2} - \mathcal{P}(X_{t+2}|X_{t+1})\} &= \mathsf{E}[(X_{t+2} - \rho_1 X_{t+1})^2] \\ &= \gamma_0(1+\rho_1^2) - 2\rho_1\gamma_1 \\ &= \mathsf{E}[(X_t - \rho_1 X_{t+1})^2] \\ &= Var[X_t - \mathcal{P}(X_t|X_{t+1})]. \end{aligned}$$

*With this we get for the partial autocorrelation of 2nd order*

$$\begin{aligned} \phi_{22} &= Corr\{X_t - \mathcal{P}(X_t|X_{t+1}), X_{t+2} - \mathcal{P}(X_{t+2}|X_{t+1})\} \\ &= \frac{Cov\{X_t - \mathcal{P}(X_t|X_{t+1}), X_{t+2} - \mathcal{P}(X_{t+2}|X_{t+1})\}}{\sqrt{Var\{X_{t+2} - \mathcal{P}(X_{t+2}|X_{t+1})\}}\sqrt{Var(X_t - \mathcal{P}(X_t|X_{t+1}))}} \\ &= \frac{\gamma_2 - 2\rho_1\gamma_1 + \rho_1^2\gamma_0}{\gamma_0(1+\rho_1^2) - 2\gamma_1\rho_1} \\ &= \frac{\rho_2 - \rho_1^2}{1-\rho_1^2} \end{aligned}$$

*which corresponds to the results in (12.18). For the AR(1) process it holds that* $\rho_2 = \rho_1^2$ *and thus* $\phi_{22} = 0$.

It holds in general for AR($p$) processes that $\phi_{kk} = 0$ for all $k > p$. In Figure 12.5 the PACF of an AR(2) process is displayed using the parameters as in Figure 12.4.

**Example 12.4 (MA(1))**
*For a MA(1) process* $X_t = \beta\varepsilon_{t-1} + \varepsilon_t$ *with* $Var(\varepsilon_t) = \sigma^2$ *it holds that* $\gamma_0 = \sigma^2(1+\beta^2)$, $\rho_1 = \beta/(1+\beta^2)$ *and* $\rho_k = 0$ *for all* $k > 1$. *For the partial autocorrelations we obtain* $\phi_{11} = \rho_1$ *and*

$$\phi_{22} = \frac{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix}} = -\frac{\rho_1^2}{1-\rho_1^2} \tag{12.19}$$

*For a MA(1) process it strictly holds that* $\phi_{22} < 0$. *If one were to continue the calculation with* $k > 2$, *one could determine that the partial autocorrelations will not reach zero.*

Figure 12.6 shows the PACF of a MA(2) process. In general for a MA($q$) process it holds that the PACF does not decay, in contrast to the autoregressive process. Compare the PACF to the ACF in Figure 12.2. This is thus a possible criterion for the specification of a linear model.

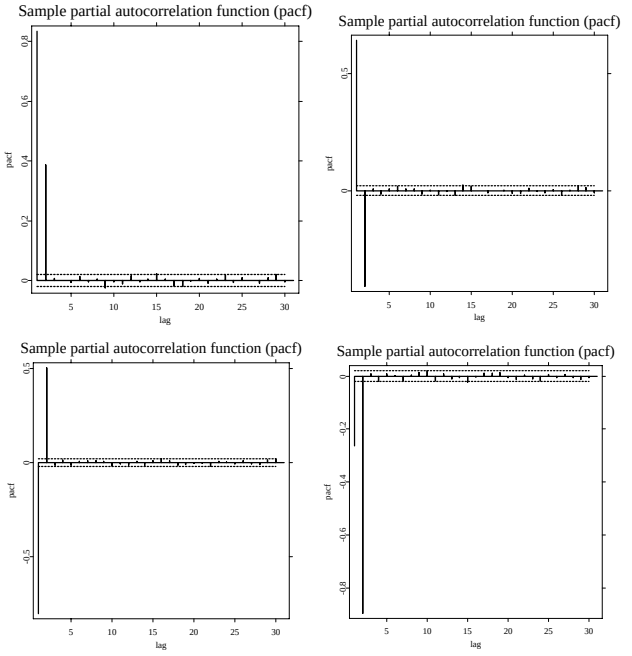


Figure 12.5: PACF of an AR(2) process with $(\alpha_1 = 0.5, \alpha_2 = 0.4)$ (top left), $(\alpha_1 = 0.9, \alpha_2 = -0.4)$ (top right), $(\alpha_1 = -0.4, \alpha_2 = 0.5)$ (bottom left) and $(\alpha_1 = -0.5, \alpha_2 = -0.9)$ (bottom right). SFEpacfar2

## 12.5 Estimation of Moments

Below we assume a stationary stochastic process $X_t$, i.e., $\mathsf{E}[X_t] = \mu$ and $\mathrm{Cov}(X_t, X_{t+\tau}) = \gamma_\tau$. In the previous sections, we have assumed that we knew the process and thus the moment that generating function was also known. In practice one observes only a realisation of the process, $X_1, \ldots, X_n$, and thus there is the problem of estimating the moment generating function.

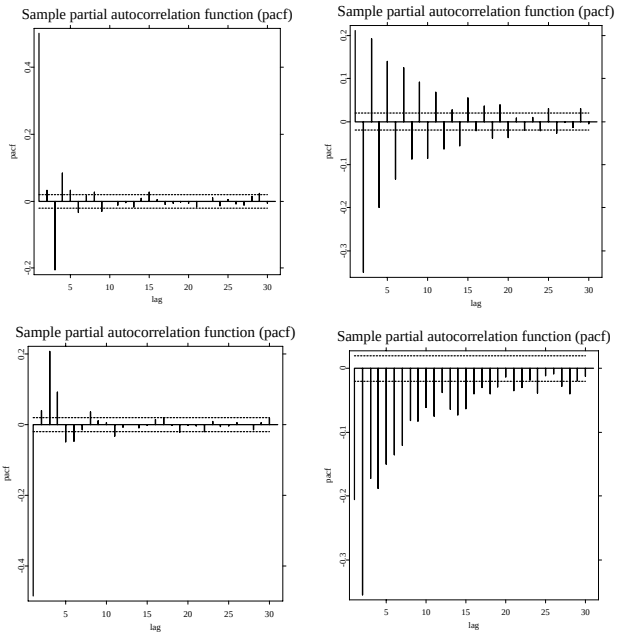


Figure 12.6: PACF of a MA(2) process with $(\beta_1 = 0.5, \beta_2 = 0.4)$ (top left), $(\beta_1 = 0.5, \beta_2 = -0.4)$ (top right), $(\beta_1 = -0.5, \beta_2 = 0.4)$ (bottom left) and $(\beta_1 = -0.5, \beta_2 = -0.4)$ (bottom right). SFEpacfma2

### 12.5.1 Estimation of the Mean Function

The parameter $\mu = \mathsf{E}[X_t]$ can be estimated with the simple arithmetic sample mean:

$$\bar{X}_n = 1/n \sum_{i=1}^{n} X_i. \tag{12.20}$$

The estimator $\bar{X}_n$ is unbiased since it holds that $\mathsf{E}[\bar{X}_n] = \mu$, and its variance is

$$\begin{aligned}
\operatorname{Var}(\bar{X}_n) &= \operatorname{Var}(1/n \sum_{i=1}^{n} X_i) \\
&= 1/n^2 \sum_{t=1}^{n} \sum_{s=1}^{n} \operatorname{Cov}(X_t, X_s) \\
&= 1/n^2 \sum_{t=1}^{n} \sum_{s=1}^{n} \gamma_{t-s} \\
&= 1/n \sum_{\tau=-(n-1)}^{n-1} \frac{n - |\tau|}{n} \gamma_\tau
\end{aligned}$$

When the autocovariance function $\gamma_\tau$ is absolutely summable, it holds that $\operatorname{Var}(\bar{X}_n) < \infty$ and $\lim_{n\to\infty} \operatorname{Var}(\bar{X}_n) = 0$. The estimator $X$ is then also a consistent estimator for $\mu$. In many cases there are more efficient estimators which take advantage of the correlation structure of the process.

The asymptotic variance

$$\lim_{n\to\infty} n \operatorname{Var}(\bar{X}_n) = \gamma_0 + 2 \sum_{\tau=1}^{\infty} \gamma_\tau$$

is denoted as $f(0)$, since this is exactly the spectral density at frequency zero. Under the absolute summability of $\gamma_\tau$ the following asymptotic distribution for the estimator holds:

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{\mathcal{L}} \mathrm{N}(0, f(0)). \tag{12.21}$$

### 12.5.2 Estimation of the Covariance Function

A possible estimator of the covariance function $\gamma_\tau$ is

$$\hat{\gamma}_{\tau,n} = \frac{1}{n} \sum_{t=1}^{n-\tau} (X_t - \bar{X}_n)(X_{t+\tau} - \bar{X}_n) \tag{12.22}$$

with the mean estimator $\bar{X}_n$ from (12.20). Instead of dividing by $n$ in (12.22) one could also divide by $n - \tau$, although the estimator would then have less favorable properties. The estimator $\hat{\gamma}_{\tau,n}$ is no longer unbiased, since the following can be shown.

$$\mathsf{E}[\hat{\gamma}_{\tau,n}] = \left(1 - \frac{\tau}{n}\right) \gamma_\tau - \left(1 - \frac{\tau}{n}\right) \operatorname{Var}(\bar{X}_n) + \mathcal{O}(n^{-2}).$$

Positive autocovariances are in general underestimated with $\hat{\gamma}_{\tau,n}$. Asymptotically $\hat{\gamma}_{\tau,n}$ is nevertheless unbiased: $\lim_{n\to\infty} E[\hat{\gamma}_{\tau,n}] = \gamma_\tau$. For the variance when terms of higher order are ignored it holds that:

$$\mathrm{Var}(\hat{\gamma}_{\tau,n}) = \frac{1}{n}\sum_{j=-\infty}^{\infty}(\gamma_j^2 + \gamma_{j-\tau}\gamma_{j+\tau}) + \mathcal{O}(n^{-1}) = \frac{1}{n}\sigma_{\tau,\infty}^2 + \mathcal{O}(n^{-1})$$

and since $\lim_{n\to\infty} \mathrm{Var}(\hat{\gamma}_{\tau,n}) = 0$ holds, $\hat{\gamma}_{\tau,n}$ is a consistent estimator for $\gamma_\tau$. Furthermore, it can be shown that the covariance estimator behaves asymptotically like a normally distributed random variable:

$$\sqrt{n}(\hat{\gamma}_{\tau,n} - \gamma_\tau) \xrightarrow{\mathcal{L}} \mathrm{N}(0, \sigma_{\tau,\infty}^2).$$

### 12.5.3 Estimation of the ACF

An obvious estimator for the ACF $\rho_\tau$ is

$$\hat{\rho}_{\tau,n} = \frac{\hat{\gamma}_{\tau,n}}{\hat{\gamma}_{0,n}}. \tag{12.23}$$

Once again we have a bias of order $1/n$, i.e.,

$$\mathsf{E}(\hat{\rho}_{\tau,n}) = \rho_\tau + \mathcal{O}(n^{-1})$$

and $\hat{\rho}_{\tau,n}$ is asymptotically unbiased. For the variance it holds that

$$\mathrm{Var}(\hat{\rho}_{\tau,n}) = \frac{1}{n}\Sigma_{\rho,\tau\tau} + \mathcal{O}(n^{-2}).$$

The estimator $\hat{\rho}_{\tau,n}$ is consistent, since $\lim_{n\to\infty} \mathrm{Var}(\hat{\rho}_{\tau,n}) = 0$. For the asymptotic distribution of the vector $\hat{\rho}_{(k),n} = (\hat{\rho}_{1,n}, \ldots, \hat{\rho}_{k,n})^\top$ it can be shown that

$$\sqrt{n}(\hat{\rho}_{(k),n} - \rho_{(k)}) \xrightarrow{\mathcal{L}} \mathrm{N}(0, \Sigma_\rho)$$

with the covariance matrix $\Sigma_\rho$ with the typical element

$$\begin{aligned}\Sigma_{\rho,kl} &= \sum_{j=-\infty}^{\infty}\rho_j\rho_{j+k+l} + \sum_{j=-\infty}^{\infty}\rho_j\rho_{j+k-l} \\ &+ 2\rho_k\rho_l\sum_{j=-\infty}^{\infty}\rho_j^2 - 2\rho_l\sum_{j=-\infty}^{\infty}\rho_j\rho_{j+k} - 2\rho_k\sum_{j=-\infty}^{\infty}\rho_j\rho_{j+l}.\end{aligned}$$

In particular for the asymptotic variance of $\sqrt{n}(\hat{\rho}_{\tau,n} - \rho_\tau)$, it holds that

$$\begin{aligned}\Sigma_{\rho,\tau\tau} &= \sum_{j=-\infty}^{\infty} \rho_j \rho_{j+2\tau} + \sum_{j=-\infty}^{\infty} \rho_j^2 \\ &+ 2\rho_\tau^2 \sum_{j=-\infty}^{\infty} \rho_j^2 - 4\rho_\tau \sum_{j=-\infty}^{\infty} \rho_j \rho_{j+\tau}.\end{aligned}$$

**Example 12.5 (MA(q))**
*For the MA(q) process in (12.1) we know that $\rho_\tau = 0$ for all $\tau > q$. Thus the asymptotic variance can be simplified from $\sqrt{n}(\hat{\rho}_{\tau,n} - \rho_\tau)$ for $\tau > q$ to*

$$\Sigma_{\rho,\tau\tau} = 1 + 2\sum_{i=1}^{q} \rho_i^2.$$

**Example 12.6 (white noise)**
*If $X_t$ is white noise, it holds that*

$$\mathsf{E}(\hat{\rho}_{\tau,n}) = -\frac{1}{n} + \mathcal{O}(n^{-2})$$

*and*

$$Var(\hat{\rho}_{\tau,n}) = \frac{1}{n} + \mathcal{O}(n^{-2})$$

*for $\tau \neq 0$. The asymptotic covariance matrix of $\sqrt{n}(\hat{\rho}_{(k),n} - \rho_{(k)})$ is the identity matrix. Using this we can build approximately 95% confidence intervals for the ACF: $[-\frac{1}{n} \pm \frac{2}{\sqrt{n}}]$.*

## 12.6 Portmanteau Statistics

With the help of the knowledge about the asymptotic distribution of the autocorrelations we can derive a statistic to test the hypothesis of white noise. One can either test the original series $X_t$ or the residuals of an ARMA$(p, q)$ process. The number of estimated parameters is in the first case $k = 0$ and in the second case $k = p + q$.

Under the null hypothesis it holds for every $m$

$$\rho_1 = 0, \ldots, \rho_m = 0.$$

The alternative hypothesis is accordingly, that at least one $\rho_i$, $1 \leq i \leq m$ is not equal to zero. Under the null hypothesis $\sqrt{n}\hat{\rho}_{\tau,n}$ is asymptotically standard normally distributed. The statistic

$$Q_m = n \sum_{j=1}^{m} \hat{\rho}_{j,n}^2$$

has an asymptotic $\chi^2$ distribution with $m-k$ degrees of freedom. One would reject the null hypothesis at a significance level of $\alpha$, as long as $Q_m > \chi^2_{m-k;\alpha}$, the $(1-\alpha)$-quantile of the Chi-squared distribution with $m-k$ degrees of freedom.

Studies show that $Q_m$ in small samples poorly approximates the asymptotic distribution. This results from the fact that $\hat{\rho}_{\tau,n}$ is a biased estimator for $\rho_\tau$. The bias is stronger for small $\tau$, and thus an asymptotically equivalent statistic can be defined as

$$Q_m^* = n(n+2) \sum_{j=1}^{m} \frac{1}{n-j} \hat{\rho}_{j,n}^2$$

which weights the empirical autocorrelations of smaller order less than those of larger order. The modified Portmanteau statistic $Q_m^*$ is therefore in small samples frequently closer to the asymptotic $\chi^2$ distribution. For large $n$, both statistics performs equally well.

## 12.7 Estimation of AR($p$) Models

A simple way to estimate the parameters of the autoregressive model

$$X_t = \alpha_1 X_{t-1} + \ldots + \alpha_p X_{t-p} + \varepsilon_t$$

with $\mathrm{Var}(\varepsilon_t) = \sigma^2$, is to use the Yule-Walker equations from (12.10), where the theoretical autocorrelation is replaced with the empirical:

$$\begin{pmatrix} 1 & \hat{\rho}_1 & \cdots & \hat{\rho}_{p-1} \\ \hat{\rho}_1 & 1 & \cdots & \hat{\rho}_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\rho}_{p-1} & \hat{\rho}_{p-2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \vdots \\ \hat{\alpha}_p \end{pmatrix} = \begin{pmatrix} \hat{\rho}_1 \\ \hat{\rho}_2 \\ \vdots \\ \hat{\rho}_p \end{pmatrix}.$$

Solving for $\hat{\alpha}$ gives the Yule-Walker estimator. It is consistent and has an asymptotic normal distribution with covariance matrix $\sigma^2\Gamma^{-1}$,

$$\Gamma = \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{p-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{p-1} & \gamma_{p-2} & \cdots & \gamma_0 \end{pmatrix}, \tag{12.24}$$

The Yule-Walker estimators are asymptotically equivalent to other estimators such as the least squares estimator, in the special case of normally distributed $\varepsilon_t$ and the maximum likelihood estimator for the normally distributed $X_t$. In this case, these estimators are also asymptotically efficient.

## 12.8 Estimation of MA($q$) and ARMA($p, q$) Models

As soon as *moving average* coefficients are included in an estimated model, the estimation turns out to be more difficult. Consider the example of a simple MA(1) model:

$$X_t = \beta\varepsilon_{t-1} + \varepsilon_t \tag{12.25}$$

with $|\beta| < 1$ and $\mathrm{Var}(\varepsilon_t) = \sigma^2$. A simple estimator for the parameter $\beta$ is obtained from the Yule-Walker equations $\gamma_0 = \sigma^2(1+\beta^2)$ and $\gamma_1 = \beta\sigma^2$. By dividing both equations we get $\rho_1 = \gamma_1/\gamma_0 = \beta/(1+\beta^2)$ and the solution to the quadratic equation is:

$$\beta = \frac{1}{2\rho_1} \pm \sqrt{\frac{1}{4\rho_1^2} - 1}. \tag{12.26}$$

The Yule-Walker estimator replaces in (12.26) the theoretical autocorrelation of 1st order $\rho_1$ with the estimator $\hat{\rho}_1$. The estimator is quite simple, but has the disadvantage that it is asymptotically inefficient.

The least squares estimator leads to non-linear systems of equations that can only be solved with iterative numerical algorithms. Using the example of a MA(1) process (12.25) this is illustrated: The LS estimator is defined by

$$\hat{\beta} = \arg\min_{\beta} \sum_{t=2}^{n} \varepsilon_t^2 = \arg\min_{\beta} \sum_{t=2}^{n} (X_t - \beta\varepsilon_{t-1})^2 \tag{12.27}$$

Given that $\varepsilon_t$ is not observed, one must turn to the AR($\infty$) representation of the MA(1) process in order to find the solution, i.e.,

$$\varepsilon_t = X_t + \sum_{k=1}^{\infty} (-\beta)^k X_{t-k}. \tag{12.28}$$

Given $X_1, \ldots, X_n$, (12.28) can be approximated by

$$\varepsilon_t = X_t + \sum_{k=1}^{t-1} (-\beta)^k X_{t-k}.$$

Solving the first order conditions

$$\frac{\partial}{\partial \beta} \sum_{t=2}^{n} \varepsilon_t^2 = 0,$$

we obtain a non-linear equation for $\beta$, which cannot be explicitly solved. For the minimisation problem (12.27) one usually implements numerical optimisation methods. The least squares estimator is asymptotically efficient and has asymptotically the same properties as the maximum likelihood (ML) estimator.

In the following we assume a stationary and invertible ARMA($p, q$) process with the AR($\infty$) representation

$$X_t = \sum_{j=1}^{\infty} \pi_j X_{t-j} + \varepsilon_t.$$

Maximum likelihood estimation allude to the distribution assumptions

$$\varepsilon_t \sim \mathrm{N}(0, \sigma^2),$$

under which $X = (X_1, \ldots, X_n)^\top$ have multivariate normal distributions with a density

$$p(x \mid \theta) = (2\pi\sigma^2)^{-n/2} |\Gamma|^{-1/2} \exp\left( -\frac{1}{2\sigma^2} x^\top \Gamma^{-1} x \right)$$

with covariance matrix $\Gamma$, which is given in (12.24), and the parameter vector

$$\theta = (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q; \sigma^2)^\top.$$

The likelihood function $L$ is then a density function interpreted as a function of the parameter vector $\theta$ for given observations, i.e., $L(\theta \mid x) = p(x \mid \theta)$.

One chooses the respective parameter vector that maximizes the likelihood function for the given observations, i.e., the ML estimator is defined by

$$\hat{\theta} = \arg\max_{\theta \in \Theta} L(\theta \mid x).$$

Under the assumption of the normal distribution the logarithm of the likelihood function

$$\log L(\theta \mid x) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2}\log|\Gamma| - \frac{1}{2\sigma^2}x^\top \Gamma^{-1} x \tag{12.29}$$

takes on a simple form without changing the maximizer $\hat{\theta}$. The log-likelihood function (12.29) is also called the *exact* log-likelihood function. One notices that, in particular, the calculation of the inverse and the determinant of the $(n \times n)$ matrix $\Gamma$ is quite involved for long time series. Therefore one often forms an approximation to the exact likelihood, which are good for long time series. One possibility is use the *conditional* distribution $p(X_t \mid X_{t-1}, \ldots, X_1; \theta)$:

$$L(\theta \mid x) = \prod_{t=1}^{n} p(X_t \mid X_{t-1}, \ldots, X_1; \theta)$$

Under the assumption of normal distributions the conditional distributions are normal with an expected value

$$\mathsf{E}[X_t \mid X_{t-1}, \ldots, X_1]$$

and variance

$$\mathrm{Var}(X_t \mid X_{t-1}, \ldots, X_1).$$

The larger $t$ is, the better the approximation of

$$\mathsf{E}[X_t \mid X_{t-1}, \ldots, X_1, \ldots] = \sum_{j=1}^{\infty} \pi_j X_{t-j}$$

by $\sum_{j=1}^{t-1} \pi_j X_{t-j}$ becomes. The conditional log-likelihood function

$$\log L^b(\theta \mid x) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{t=1}^{n}(X_t - \sum_{j=1}^{t-1}\pi_j X_{t-j})^2 \tag{12.30}$$

can be calculated from the data $X_1, \ldots, X_n$ and optimised with respect to the parameter $\theta$. As an initial value for the numerical optimisation algorithm the Yule-Walker estimators, for example, can be used (except in specific cases of asymptotic inefficiency).

To compare the exact and the conditional likelihood estimators consider a MA(1) process (12.25) with $\beta = 0.5$ and $\varepsilon_t \sim \mathrm{N}(0, 1)$. The matrix $\Gamma$ is band diagonal with elements $1 + \beta^2$ on the main diagonal and $\beta$ on diagonals both above and below it. Two realisations of the process with $n = 10$ and $n = 20$ are shown in Figure 12.7. Since the process has only one parameter, one can simply search in the region (-1,1). This is shown for both estimators in Figure 12.8 ($n = 10$) and 12.9 ($n = 20$). For the process with $n = 10$ one still sees a clear discrepancy between both likelihood functions, which for $n = 20$ can be ignored. Both estimators are in this case quite close to the true parameter 0.5.

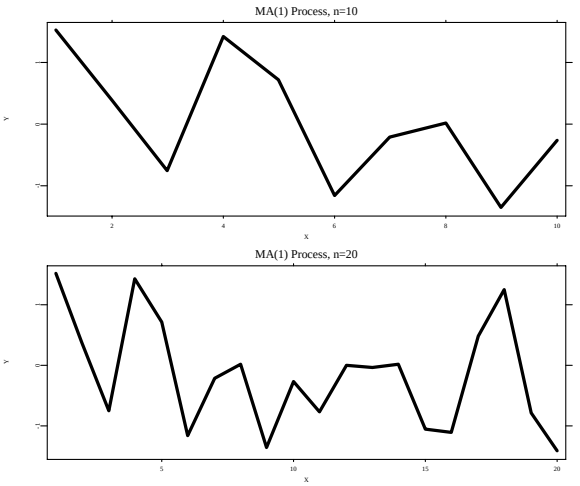


Figure 12.7: Two realisations of a MA(1) process with $\beta = 0.5$, $\varepsilon_t \sim \mathrm{N}(0, 1)$, $n = 10$ (above) and $n = 20$ (below). SFEplotma1

Under some technical assumptions the ML estimators are consistent, asymptotically efficient and have an asymptotic normal distribution:

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{\mathcal{L}} \mathrm{N}(0, J^{-1})$$

with the Fisher Information matrix

$$J = \mathsf{E}\left[-\frac{\partial^2 \log L(\theta, x)}{\partial\theta\partial\theta^\top}\right]. \tag{12.31}$$

For the optimisation of the likelihood function one frequently uses numerical

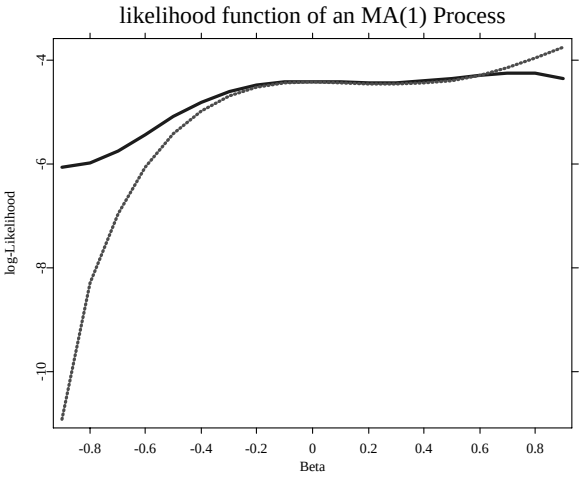


Figure 12.8: Exact (solid) and conditional (dashed) likelihood functions for the MA(1) process from figure 12.7 with $n = 10$. The true parameter is $\beta = 0.5$. SFElikma1

methods. The necessary condition for a maximum is

$$\text{grad } l^b(\theta) = 0$$

with $l^b = \log L(\theta \mid x)$. By choosing an initial value $\theta_0$ (for example, the Yule-Walker estimator), and the Taylor approximation

$$\text{grad } l^b(\theta) \approx \text{grad } l^b(\theta_0) + \text{Hess } l^b(\theta_0)(\theta - \theta_0)$$

one obtains the following relation:

$$\theta = \theta_0 - \text{Hess } l^b(\theta_0)^{-1}\text{grad } l^b(\theta_0).$$

Since generally one does not immediately hit the maximising parameter, one builds the iteration

$$\theta_{j+1} = \theta_j - \text{Hess } l^b(\theta_j)^{-1}\text{grad } l^b(\theta_j)$$

with $j = 1, 2, \ldots$ until a convergence is reached, i.e., $\theta_{j+1} \approx \theta_j$. Often it is easier to use the expectation of the Hessian matrix, that is, the information matrix from (12.31):

$$\theta_{j+1} = \theta_j + J(\theta_j)^{-1}\text{grad } l^b(\theta_j). \tag{12.32}$$

The notation $J(\theta_j)$ here means that (12.31) is evaluated at $\theta_j$. The iteration (12.32) is called the *score*-algorithm or Fisher scoring.

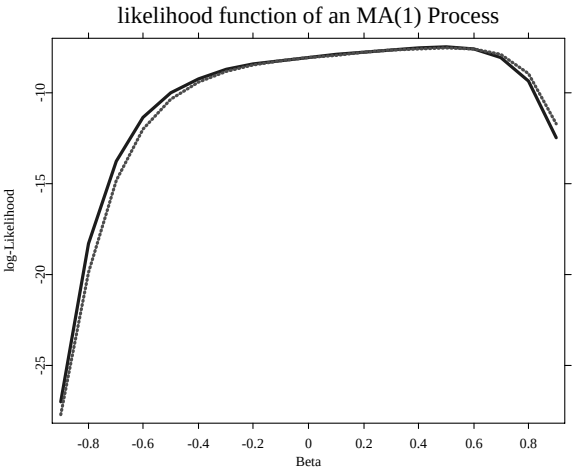


Figure 12.9: Exact (solid) and conditional (dashed) likelihood functions for the MA(1) process from figure 12.7 with $n = 20$. The true parameter is $\beta = 0.5$. SFElikma1

## 12.9 Recommended Literature

Comprehensive textbooks on classic time series analysis are, for example, Schlittgen and Streitberg (1995), Brockwell and Davis (1991), Gouriéroux and Monfort (1996). In addition, classic books are Box and Jenkins (1976) and Hamilton (1994).

# 13 Time Series with Stochastic Volatility

In the previous chapters we have already discussed that volatility plays an important role in modelling financial systems and time series. Unlike the term structure, volatility is unobservable and thus must be estimated from the data.

Reliable estimations and forecasts of volatility are important for large credit institutes where volatility is directly used to measure risk. The risk premium, for example, is often specified as a function of volatility. It is interesting to find an appropriate model for volatility. The capability of macroeconomic factors to forecast volatility has already been examined in the literature. Although macroeconomic factors have some forecasting capabilities, the most important factor seems to be the lagged endogenous return. As a result recent studies are mainly concentrated on time series models.

Stock, exchange rates, interest rates and other financial time series have *stylized facts* that are different from other time series. A good candidate for modelling financial time series should represent the properties of stochastic processes. Neither the classic linear AR or ARMA processes nor the nonlinear generalisations can fulfil this task. In this chapter we will describe the most popular volatility class of models: the ARCH (*autoregressive conditional heteroscedasticity*) model that can replicate these stylised facts appropriately.

**Stylised fact 1:** *Time series of share prices $X_t$ and other basic financial instruments are not stationary time series and possess a local trend at least.*

Similar to the ARIMA model in Chapter 12, we transform the original data by first taking differences to get a stationary time series. Here we consider the log return (see Definition 11.15) instead of the original share prices. We simply call it *return* in this chapter. One could consider the simple return $R_t$ as well (see Definition 11.14).

**Stylised fact 2:** *Returns $r_t$ have a leptokurtic distribution. The empirically estimated kurtosis is mostly greater than 3.*

We have discussed the properties of the return's distribution in Section 3.3

and Section 11.2. The leptokurtosis can be illustrated in a comparison of the density of a normal distribution and a kernel estimator of the adjusted data (see Figure 15.1). We can see in Theorem 13.3 that an ARCH process has a kurtosis greater than 3 even if the innovation of the process itself is normally distributed.

**Stylised fact 3:** *The return process is white noise (Definition 11.8) since the sample autocorrelation $\hat{\rho}_{\tau,n}, k \neq 0$ (12.23) is not significantly different from 0. Furthermore the white noise is not independent since the sample auto-correlations of squared and absolute returns are clearly greater than 0.*

ARCH models possess the characteristic (Theorem 13.1) that we have already described in Section 11.2. A stronger condition than pairwise uncorrelation of returns is that returns are unpredictable, which is connected to the no arbitrage condition. As in Section 11.3 $\mathcal{F}_t$ denotes the information set at time t. The best prediction of return $r_{t+1}$ at day $t$ for day $t+1$ is the conditional expectation $r_{t+1|t} = \mathsf{E}[r_{t+1}|\mathcal{F}_t]$ (Theorem 11.1) based on the information set $\mathcal{F}_t$. The time series of the return is called unpredictable if

$$r_{t+1|t} = \mathsf{E}[r_{t+1}|\mathcal{F}_t] = \mathsf{E}[r_{t+1}],$$

i.e. the best prediction of the next return is simply its unconditional mean. The information set $\mathcal{F}_t$ gives no hints for predicting future prices. ARCH processes are automatically unpredictable (Definition 13.1).

An unpredictable time series is always white noise because the auto-correlation is equal to 0. It is even possible that a linear prediction is better than the expectation estimated only by the unpredictable time series (see the proof of Theorem 13.1). The condition of unpredictability is actually stronger than pairwise uncorrelation. A predictable white noise is, for example $\varepsilon_t = \eta_t + \gamma\eta_{t-1}\eta_{t-2}$, where $\eta_t$ is independent white noise with expectation of 0. This bilinear process has vanishing auto-correlations but $\mathsf{E}[\varepsilon_{t+1}|\mathcal{F}_t] = \gamma\eta_t\eta_{t-1} \neq 0 = \mathsf{E}[\varepsilon_{t+1}]$.

If the returns were predictable we could develop a trading strategy based on the resulting predictions of price, which would give us a positive profit. The existence of a stochastic arbitrage probability obviously contradicts the assumption of a perfect financial market (Section 2.1).

**Stylised fact 4:** *Volatility tends to form clusters: After a large (small) price change (positive or negative) a large (small) price change tends to occur. This effect is called volatility clustering.*

We will consider the properties of financial time series in more detail in the following section. According to the stylised fact 4, the squared returns are

positively correlated. Thus returns are conditionally heteroscedastic i.e.

$$\mathrm{Var}[r_{t+1}|\mathcal{F}_t] \neq \mathrm{Var}[r_{t+1}].$$

The returns $r_t$ are not independent but their variability depends on recent changes of price.

## 13.1 ARCH and GARCH Models

Following the introduction of ARCH models in the eighties there were enormous theoretical and practical developments in financial econometrics. It became clear that ARCH models could efficiently and quite easily represent the typical empirical findings in financial time series, e.g. the conditional heteroscedasticity. In particular after the collapse of the Bretton Woods system and the implementation of flexible exchange rates in the seventies ARCH models became increasingly used by researchers and practitioners.

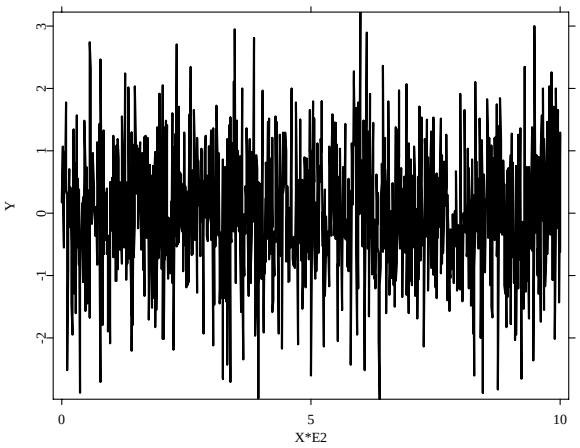


Figure 13.1: Normally distributed white noise. SFEtimewr

In addition a far-reaching agreement was formed, that returns cannot be regarded as i.i.d. and at most as being uncorrelated. This argument holds at least for financial time series of relatively high frequency, for example for daily data. In Figure 13.1 we show a normally distributed white noise, a GARCH(1,1) process in Figure 13.2 and the DAFOX index (1993-96) in Figure 13.3, see http://finance.wiwi.uni-karlsruhe.de/Forschung/dafox.html.

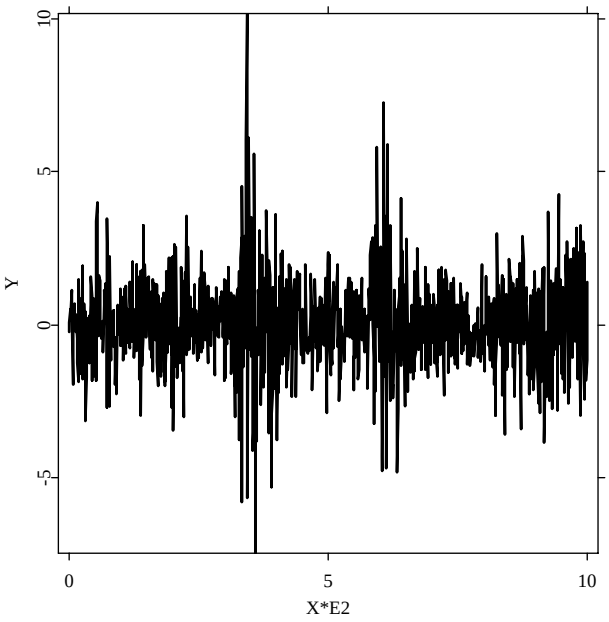


Figure 13.2: A GARCH(1,1) process ($\alpha = 0.15$, $\beta = 0.8$). SFEtimegarc

It can be seen from the figure that the GARCH process is obviously more appropriate for modelling stock returns than white noise.

However the ARCH model is only the starting point of the empirical study and relies on a wide range of specification tests. Some practically relevant disadvantages of the ARCH model have been discovered recently, for example, the definition and modelling of the persistence of shocks and the problem of modelling asymmetries. Thus a large number of extensions of the standard ARCH model have been suggested. We will discuss them in detail later.

Let $X_t$ be a discrete stochastic process and from Definition 11.15 $r_t = \log X_t / X_{t-1}$ the relative increase or the *return* of the process $X_t$. If the returns are almost independent and identically distributed, then $X_t$ follows a geometric random walk. It is assumed in ARCH models that the returns depend on past information with a specific form.

As mentioned before $\mathcal{F}_t$ denotes the information set at time $t$, which encompasses $X_t$ and all the past realisations of the process $X_t$. This means in a

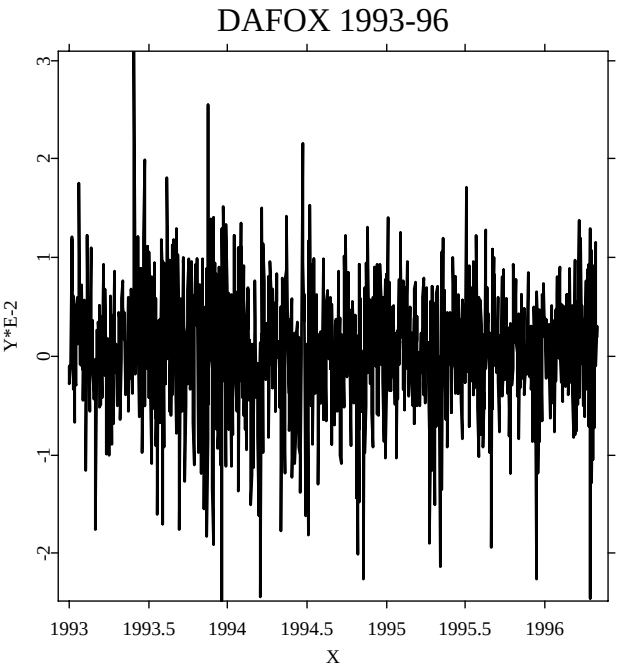


Figure 13.3: DAFOX returns from 1993 to 1996. SFEtimedax

general model

$$r_t = \mu_t + \varepsilon_t \tag{13.1}$$

with $\mathsf{E}[\varepsilon_t \mid \mathcal{F}_{t-1}] = 0$. Here $\mu_t$ can represent the risk premium which results from the econometric models and is time dependent. The stochastic error term $\varepsilon_t$ is no longer independent but centred and uncorrelated. In ARCH models the conditional variance of $\varepsilon_t$ is a linear function of the lagged squared error terms.

### 13.1.1 ARCH(1): Definition and Properties

The ARCH model of order 1, ARCH(1), is defined as follows:

**Definition 13.1 (ARCH(1))**
*The process $\varepsilon_t$, $t \in \mathbb{Z}$, is ARCH(1), if* $\mathsf{E}[\varepsilon_t \mid \mathcal{F}_{t-1}] = 0$,

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 \tag{13.2}$$

*with* $\omega > 0$, $\alpha \geq 0$ *and*

- $Var(\varepsilon_t \mid \mathcal{F}_{t-1}) = \sigma_t^2$ *and* $Z_t = \varepsilon_t/\sigma_t$ *is i.i.d. (strong ARCH)*
- $Var(\varepsilon_t \mid \mathcal{F}_{t-1}) = \sigma_t^2$ *(semi-strong ARCH),*
- $\mathcal{P}(\varepsilon_t^2 \mid 1, \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots, \varepsilon_{t-1}^2, \varepsilon_{t-2}^2, \ldots) = \sigma_t^2$ *(weak ARCH),*

where $\mathcal{P}$ is the best linear projection described in Section 12.4. Obviously a strong ARCH(1) process is also semi-strong and a semi-strong also weak. On the other hand the conditional variance of a weak ARCH(1) process can be non-linear (unequal to $\sigma_t^2$). In this case it cannot be a semi-strong ARCH process.

Setting $Z_t = \varepsilon_t/\sigma_t$, it holds for the semi-strong and the strong ARCH models that $\mathsf{E}[Z_t] = 0$ and $\mathrm{Var}(Z_t) = 1$. In strong ARCH models $Z_t$ is i.i.d. so that no dependence can be modelled in higher moments than the second moment. It is frequently assumed that $Z_t$ is normally distributed, which means $\varepsilon_t$ is conditionally normally distributed:

$$\varepsilon_t | \mathcal{F}_{t-1} \sim \mathrm{N}(0, \sigma_t^2). \tag{13.3}$$

Under (13.3) the difference between the strong and the semi-strong ARCH models disappears.

Originally only strong and semi-strong ARCH models are discussed in the literature. Weak ARCH models are important because they are closed under temporal aggregation. If, for example, daily returns follow a weak ARCH process, then the weekly and monthly returns are also weak ARCH with corresponding parameter adjustments. This phenomenon holds in general for strong and semi-strong models.

According to Definition 13.1 the process $\varepsilon_t$ is a martingale difference and therefore white noise.

**Theorem 13.1**
*Assume that the process* $\varepsilon_t$ *is a weak ARCH(1) process with* $Var(\varepsilon_t) = \sigma^2 < \infty$. *Then it follows that* $\varepsilon_t$ *is white noise.*

**Proof:**
From $\mathsf{E}[\varepsilon_t \mid \mathcal{F}_{t-1}] = 0$ it follows that $\mathsf{E}[\varepsilon_t] = 0$ and $\mathrm{Cov}(\varepsilon_t, \varepsilon_{t-k}) = \mathsf{E}[\varepsilon_t \varepsilon_{t-k}] = \mathsf{E}[\mathsf{E}(\varepsilon_t \varepsilon_{t-k} \mid \mathcal{F}_{t-1})] = \mathsf{E}[\varepsilon_{t-k}\, \mathsf{E}(\varepsilon_t \mid \mathcal{F}_{t-1})] = 0$. □
Note that $\varepsilon_t$ is not an independent white noise.

**Theorem 13.2 (Unconditional variance of the ARCH(1))**
*Assume the process* $\varepsilon_t$ *is a semi-strong ARCH(1) process with* $Var(\varepsilon_t) = \sigma^2 <$

$\infty$. *Then it holds that*

$$\sigma^2 = \frac{\omega}{1-\alpha}.$$

**Proof:**
$\sigma^2 = \mathsf{E}[\varepsilon_t^2] = \mathsf{E}[\mathsf{E}(\varepsilon_t^2 \mid \mathcal{F}_{t-1})] = \mathsf{E}[\sigma_t^2] = \omega + \alpha\,\mathsf{E}[\varepsilon_{t-1}^2] = \omega + \alpha\sigma^2$. It holds then $\sigma^2 = \omega/(1-\alpha)$ when $\alpha < 1$. □
$\alpha < 1$ is the necessary and sufficient condition for a weak stationarity of a semi-strong process.

If the innovation $Z_t = \varepsilon_t/\sigma_t$ is symmetrically distributed around zero, then all odd moments of $\varepsilon_t$ are equal to zero. Under the assumption of normal distribution (13.3) the conditions for the existence of higher even moments can be derived.

**Theorem 13.3 (Fourth Moment)**
*Let $\varepsilon_t$ be a strong ARCH(1) process, $Z_t \sim N(0,1)$ and $\mathsf{E}[\varepsilon_t^4] = c < \infty$. Then*

1. $$\mathsf{E}[\varepsilon_t^4] = \frac{3\omega^2}{(1-\alpha)^2}\frac{1-\alpha^2}{1-3\alpha^2}$$
   *with $3\alpha^2 < 1$.*
2. *the unconditional distribution of $\varepsilon_t$ is leptokurtic.*

**Proof:**

1. $c = \mathsf{E}[\varepsilon_t^4] = \mathsf{E}[\mathsf{E}(\varepsilon_t^4 \mid \mathcal{F}_{t-1})] = \mathsf{E}[\sigma_t^4\,\mathsf{E}(Z_t^4 \mid \mathcal{F}_{t-1})] = \mathsf{E}[Z_t^4]\,\mathsf{E}[(\omega + \alpha_1\varepsilon_{t-1}^2)^2] = 3(\omega^2 + 2\omega\alpha\,\mathsf{E}[\varepsilon_{t-1}^2] + \alpha^2\,\mathsf{E}[\varepsilon_{t-1}^4])$. Since $\mathsf{E}[\varepsilon_{t-1}^2] = \omega/(1-\alpha)$ and $\mathsf{E}[\varepsilon_{t-1}^4] = c$, after rearranging the claim follows.
2. $$\mathrm{Kurt}(\varepsilon_t) = \frac{\mathsf{E}[\varepsilon_t^4]}{\mathsf{E}[\varepsilon_t^2]^2} = 3\frac{1-\alpha^2}{1-3\alpha^2} \geq 3.$$

□

For the boundary case $\alpha = 0$ and the normally distributed innovations $\mathrm{Kurt}(\varepsilon_t) = 3$, while for $\alpha > 0$ it holds that $\mathrm{Kurt}(\varepsilon_t) > 3$. The unconditional distribution is also leptokurtic under conditional heteroscedasticity, i.e., the curvature is high in the middle of the distribution and the tails are fatter than those of a normal distribution, which is frequently observed in financial markets.

The thickness of the tails and thus the existence of moments depend on the parameters of the ARCH models. The variance of the ARCH(1) process is finite when $\alpha < 1$ (Theorem 13.2), while the fourth moment in the case of normally distributed error terms exists when $3\alpha^2 < 1$ (Theorem 13.3). As early as the sixties Mandelbrot questioned the existence of the variance of several financial time series. Frequently empirical distributions have so fat tails that one cannot conclude a finite variance. In order to make empirical conclusions on the degree of the tail's thickness of the unconditional distribution, one can assume, for example, that the distribution is a Pareto type, i.e., for large $x$:

$$\mathrm{P}(x) = \mathrm{P}(X_t > x) \sim kx^{-a}$$

for $a > 0$. When $a > c$, it holds that $\mathsf{E}[|X_t|^c] < \infty$. The question is, how can we estimate the tail index $a$? A simple method follows from the conclusion that for large $x$ the log function $\mathrm{P}(x)$ is linear, i.e.,

$$\log \mathrm{P}(x) \approx \log k - a \log x. \tag{13.4}$$

Therefore we can build the order statistics $X_{(1)} > X_{(2)} > \ldots > X_{(n)}$ and estimate the probability $\mathrm{P}(x)$ for $x = X_{(i)}$ using the relative frequency

$$\frac{\#\{t; X_t \geq X_{(i)}\}}{n} = \frac{i}{n}.$$

In (13.4) $\mathrm{P}(X_{(i)})$ is replaced with the estimator $i/n$:

$$\log \frac{i}{n} \approx \log k - a \log X_{(i)}, \tag{13.5}$$

from which $a$ can be estimated from the regression of $i/n$ on $X_{(i)}, i = 1, ..., n$, using the least squares method. In general only a small part of the data will be used for the regression, since the linear approximation of $\log \mathrm{P}(x)$ is only appropriate in the tail. Thus only the largest order statistics are used to estimate the regression (13.5). Figure 13.4 shows the regression (13.5) for the DAFOX from 1974 to 1996 with $m = 20$, i.e. we choose the 20 largest observations. The slope of the least squares (LS) line is -3.25. It indicates that the variance and the third moment of this time series are finite whereas the fourth moment and the kurtosis are not finite.

Hill (1975) has suggested an estimator using the maximum likelihood method:

$$\hat{a} = \left( \frac{1}{m-1} \sum_{i=1}^{m} \log X_{(i)} - \log X_{(m)} \right)^{-1}, \tag{13.6}$$

where $m$ is the number of observations taken from the tail and used in the estimation. How to choose $m$ obviously raises a problem. When $m$ is too

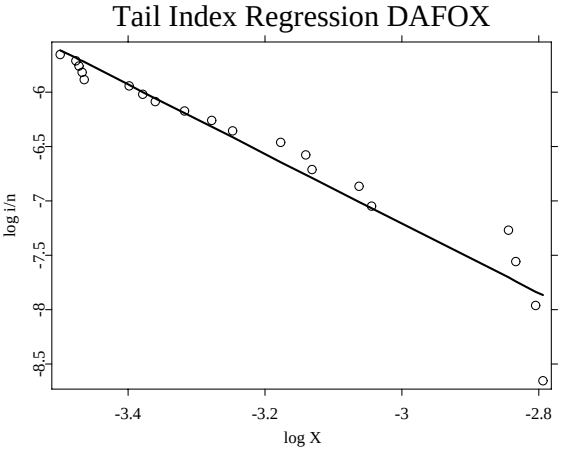


Figure 13.4: The right side of the logged empirical distribution of the DAFOX returns from 1974 to 1996. SFEtaildax

large, the approximation of the distribution is no longer good; when $m$ is too small, the bias and the variance of the estimator could increase. A simple rule of thumb says that $m/n$ should be around 0.5% or 1%. Clearly one requires a large amount of data in order to estimate $a$ well. As an example we can consider again the daily returns of German stocks from 1974 to 1996, a total of 5747 observations per stock. The results of the ordinary least squares estimator and the Hill estimator with $m = 20$ and $m = 50$ are given in Table 13.1. In every case the estimators are larger than 2, which indicates the existence of variances. The third moment may not exist in some cases, for example, for Allianz and Daimler.

**Theorem 13.4 (Representation of an ARCH(1) process)**
*Let $\varepsilon_t$ be a strong ARCH(1) process with $Var(\varepsilon_t) = \sigma^2 < \infty$. It holds that*

$$\varepsilon_t^2 = \omega \sum_{k=0}^{\infty} \alpha^k \prod_{j=0}^{k} Z_{t-j}^2$$

*and the sum converges in $L_1$.*

**Proof:**
Through the recursive substitution of $\varepsilon_s^2 = \sigma_s^2 Z_s^2$ and $\sigma_s^2 = \omega + \alpha\varepsilon_{s-1}^2$. The

convergence follows from

$$\begin{aligned} \mathsf{E}[\varepsilon_t^2 - \omega \sum_{k=0}^{m} \alpha^k \prod_{j=0}^{k} Z_{t-j}^2] &= \alpha^{m+1} \mathsf{E}[\varepsilon_{t-m-1}^2 \prod_{j=0}^{m} Z_{t-j}^2] \\ &= \alpha^{m+1} \mathsf{E}[\varepsilon_{t-m-1}^2] \longrightarrow 0 \end{aligned}$$

for $m \longrightarrow \infty$, since $Z_t$ is independent with $\mathsf{E}(Z_t^2) = 1$. □

**Theorem 13.5**
*Let $\varepsilon_t$ be a stationary strong ARCH(1) process with $\mathsf{E}(\varepsilon_t^4) = c < \infty$ and $Z_t \sim N(0,1)$. It holds that*

| | LS | | Hill | |
|---|---|---|---|---|
| $m$ | 20 | 50 | 20 | 50 |
| DAFOX | 3.25 | 2.94 | 3.17 | 2.88 |
| ALLIANZ | 2.29 | 2.44 | 2.28 | 2.26 |
| BASF | 4.19 | 4.29 | 4.58 | 4.01 |
| BAYER | 3.32 | 3.20 | 3.90 | 3.23 |
| BMW | 3.42 | 3.05 | 3.05 | 2.89 |
| COMMERZBANK | 6.58 | 4.67 | 7.14 | 5.19 |
| DAIMLER | 2.85 | 2.85 | 2.43 | 2.56 |
| DEUTSCHE BANK | 3.40 | 3.26 | 3.41 | 3.29 |
| DEGUSSA | 3.03 | 4.16 | 2.93 | 3.30 |
| DRESDNER | 5.76 | 4.08 | 4.20 | 4.32 |
| HOECHST | 4.77 | 3.68 | 5.66 | 4.05 |
| KARSTADT | 3.56 | 3.42 | 3.11 | 3.16 |
| LINDE | 3.30 | 3.35 | 3.87 | 3.37 |
| MAN | 3.83 | 3.66 | 3.17 | 3.45 |
| MANNESMANN | 3.19 | 3.85 | 2.84 | 3.22 |
| PREUSSAG | 3.52 | 4.11 | 3.57 | 3.68 |
| RWE | 3.87 | 3.78 | 3.51 | 3.54 |
| SCHERING | 3.34 | 4.82 | 3.22 | 3.64 |
| SIEMENS | 6.06 | 4.50 | 5.96 | 5.23 |
| THYSSEN | 5.31 | 5.36 | 4.67 | 4.97 |
| VOLKSWAGEN | 4.59 | 3.31 | 4.86 | 4.00 |

Table 13.1: Least Square (LS) and Hill estimators of the tail index $a$ with $m$ observations used for the estimation.

*1.*

$$\varepsilon_t^2 = \omega \sum_{k=0}^{\infty} \alpha^k \prod_{j=0}^{k} Z_{t-j}^2$$

*and the sum converges in $L_2$.*

2. $\eta_t = \sigma_t^2(Z_t^2 - 1)$ *is white noise.*
3. $\varepsilon_t^2$ *is an AR(1) process with* $\varepsilon_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \eta_t$.

**Proof:**

1. As in Theorem 13.4. The convergence is $L_2$ follows from

$$\begin{aligned}\mathsf{E}[(\varepsilon_t^2 - \omega\sum_{k=0}^{m}\alpha^k\prod_{j=0}^{k}Z_{t-j}^2)^2] &= \mathsf{E}[(\alpha^{m+1}\varepsilon_{t-m-1}^2\prod_{j=0}^{m}Z_{t-j}^2)^2] \\ &= \alpha^{2(m+1)}3^{m+1}\,\mathsf{E}[\varepsilon_{t-m-1}^4] \\ &= \alpha^{2(m+1)}3^{m+1}c \longrightarrow 0\end{aligned}$$

   for $m \longrightarrow \infty$, since $3\alpha^2 < 1$ due to the assumption that $\mathsf{E}(\varepsilon_t^4)$ is finite and since $Z_t$ is independent with $\mathrm{Kurt}(Z_t) = 3$.

2. a) $\mathsf{E}[\eta_t] = \mathsf{E}[\sigma_t^2]\,\mathsf{E}[Z_t^2 - 1] = 0$
   b)

$$\begin{aligned}\mathrm{Var}(\eta_t) &= \mathsf{E}[\sigma_t^4]\,\mathsf{E}[(Z_t^2-1)^2] = 2\,\mathsf{E}[(\omega + \alpha\varepsilon_{t-1}^2)^2] \\ &= 2(\omega^2 + 2\alpha\omega\,\mathsf{E}[\varepsilon_{t-1}^2] + \alpha^2\,\mathsf{E}[\varepsilon_{t-1}^4]) = \text{const.}\end{aligned}$$

   is independent of $t$.
   c)

$$\begin{aligned}\mathrm{Cov}(\eta_t, \eta_{t+s}) &= \mathsf{E}[\sigma_t^2(Z_t^2-1)\sigma_{t+s}^2(Z_{t+s}^2-1)] \\ &= \mathsf{E}[\sigma_t^2(Z_t^2-1)\sigma_{t+s}^2]\,\mathsf{E}[(Z_{t+s}^2-1)] \\ &= 0 \qquad \text{for } s \neq 0.\end{aligned}$$

3. It follows from rearranging: $\varepsilon_t^2 = \sigma_t^2 Z_t^2 = \sigma_t^2 + \sigma_t^2(Z_t^2 - 1) = \omega + \alpha\varepsilon_{t-1}^2 + \eta_t$.

□

**Remark 13.1**
*Nelson (1990a) shows that the strong ARCH(1) process* $\varepsilon_t$ *is strictly stationary when* $\mathsf{E}[\log(\alpha Z_t^2)] < 0$. *If, for example,* $Z_t \sim N(0,1)$, *then the condition for strict stationarity is* $\alpha < 3.5622$, *which is weaker than the condition for covariance-stationarity with* $\alpha < 1$ *due to the assumption that the variance is finite.*

The dynamics of the volatility process in the case of ARCH(1) is essentially determined by the parameter $\alpha$. In Theorem 13.5 it was shown that the square of an ARCH(1) process follows an AR(1) process. The correlation structure of the empirical squared observations of returns are frequently more complicated than a simple AR(1) process. In Section 13.1.3 we will consider an ARCH model of order $q$ with $q > 1$, which allows a more flexible modelling of the correlation structure.

The volatility is a function of the past squared observations in ARCH models in a narrow sense. In the more general GARCH models (Section 13.1.5) it may depend on the past squared volatilities in addition. These models belong to the large group of unpredictable time series with *stochastic volatility*. In the strong form, they have $\varepsilon_t = \sigma_t Z_t$ where $\sigma_t$ is $\mathcal{F}_{t-1}$-measurable, i.e. the volatility $\sigma_t$ depends only on the information to the time point $t-1$ and the i.i.d. innovations $Z_t$ with $\mathsf{E}[Z_t] = 0, \mathrm{Var}(Z_t) = 1$. For such a time series it holds $\mathsf{E}[\varepsilon_t|\mathcal{F}_{t-1}] = 0, \mathrm{Var}(\varepsilon_t|\mathcal{F}_{t-1}) = \sigma_t^2$, i.e. $\varepsilon_t$ is unpredictable and, except in the special case that $\sigma_t \overset{\text{def}}{=} \text{const.}$, conditionally heteroscedastic. The stylised facts 2-4 are only fulfilled under certain qualitative assumptions. For example, in order to produce volatility cluster $\sigma_t$ must tend to be large when the squared observations or volatilities of the recent past observations are large. The generalisations of the ARCH models observed in this section fulfill the corresponding conditions.

**Remark 13.2**
*At first glance stochastic volatility models in discrete time deliver a different approach in modelling the financial data compared with diffusion processes, on which the Black-Schole model and its generalisation are based (Section 5.4). Nelson (1990b) has however shown that ARCH and also the more general GARCH processes converge in the limit to a diffusion process in continuous time when the difference of the time points of the successive observations goes against zero.*

*This result is often used reversely in order to estimate the parameter of financial models in the continuous time where one approximates the corresponding diffusion processes through discrete GARCH time series and estimates its parameter. Nelson (1990b) shows only the convergence of GARCH processes against diffusion processes in a weak sense (convergence on the distribution). A recent work by Wang (2002) however, shows that the approximation does not hold in a stronger sense, in particular the likelihood process is not asymptotically equivalent. In this sense the maximum likelihood estimators for the discrete time series do not converge against the parameters of the diffusion limit process.*

### 13.1.2 Estimation of ARCH(1) Models

Theorem 13.5 says that an ARCH(1) process can be represented as an AR(1) process in $X_t^2$. A simple Yule-Walker estimator uses this property:

$$\hat{\alpha}^{(0)} = \frac{\sum_{t=2}^{n}(\varepsilon_t^2 - \hat{\omega}^{(0)})(\varepsilon_{t-1}^2 - \hat{\omega}^{(0)})}{\sum_{t=2}^{n}(\varepsilon_t^2 - \hat{\omega}^{(0)})^2}$$

with $\hat{\omega}^{(0)} = n^{-1}\sum_{t=1}^{n}\varepsilon_t^2$. Since the distribution of $\varepsilon_t^2$ is naturally not normal, the Yule-Walker estimator is inefficient. However it can be used as an initial value for iterative estimation methods.

The estimation of ARCH models is normally done using the maximum likelihood (ML) method. Assuming that the returns $\varepsilon_t$ have a conditionally normal distribution, we have:

$$p(\varepsilon_t \mid \mathcal{F}_{t-1}) = \frac{1}{\sqrt{2\pi}\sigma_t}\exp\left\{-\frac{1}{2}\frac{\varepsilon_t^2}{\sigma_t^2}\right\}, \tag{13.7}$$

The log-likelihood function $l(\omega, \alpha)$ can be written as a function of the parameters $\omega$ and $\alpha$:

$$\begin{aligned} l(\omega,\alpha) &= \sum_{t=2}^{n} l_t(\omega,\alpha) + \log p_\varepsilon(\varepsilon_1) \qquad (13.8)\\ &= \sum_{t=2}^{n}\log p(\varepsilon_t \mid \mathcal{F}_{t-1}) + \log p_\varepsilon(\varepsilon_1)\\ &= -\frac{n-1}{2}\log(2\pi) - \frac{1}{2}\sum_{t=2}^{n}\log(\omega + \alpha\varepsilon_{t-1}^2)\\ &\quad -\frac{1}{2}\sum_{t=2}^{n}\frac{\varepsilon_t^2}{\omega + \alpha\varepsilon_{t-1}^2} + \log p_\varepsilon(\varepsilon_1), \end{aligned}$$

where $p_\varepsilon$ is the stationary marginal density of $\varepsilon_t$. A problem is that the analytical expression for $p_\varepsilon$ is unknown in ARCH models thus (13.8) cannot be calculated. In the conditional likelihood function $l^b = \log p(\varepsilon_n, \ldots, \varepsilon_2 \mid \varepsilon_1)$ the expression $\log p_\varepsilon(\varepsilon_1)$ disappears:

$$\begin{aligned} l^b(\omega,\alpha) &= \sum_{t=2}^{n} l_t(\omega,\alpha) \qquad (13.9)\\ &= \sum_{t=2}^{n}\log p(\varepsilon_t \mid \mathcal{F}_{t-1})\\ &= -\frac{n-1}{2}\log(2\pi) - 1/2\sum_{t=2}^{n}\log(\omega + \alpha\varepsilon_{t-1}^2) - 1/2\sum_{t=2}^{n}\frac{\varepsilon_t^2}{\omega + \alpha\varepsilon_{t-1}^2}. \end{aligned}$$

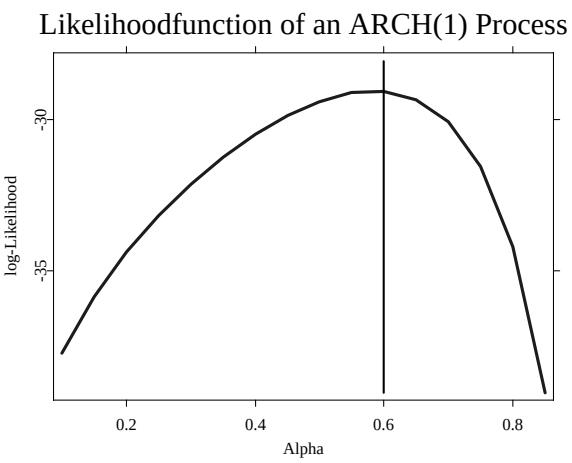


Figure 13.5: Conditional likelihood function of a generated ARCH(1) process with $n = 100$. The true parameter is $\alpha = 0.5$. SFElikarch1

For large $n$ the difference $l - l^b$ is negligible.

Figure 13.5 shows the conditional likelihood of a generated ARCH(1) process with $n = 100$. The parameter $\omega$ is chosen so that the unconditional variance is constant everywhere, i.e., with a variance of $\sigma^2$, $\omega = (1 - \alpha)\sigma^2$. The optimisation of the likelihood of an ARCH(1) model can be found by analysing the graph. Most often, we would like to know the precision of the estimator as well. Essentially it is determined by the second derivative of the likelihood at the optimisation point by the asymptotic properties of the ML estimator (see Section 13.1.6). Furthermore one has to use numerical methods such as the score algorithm introduced in Section 12.8 to estimate the parameters of the models with a larger order. In this case the first and second partial derivatives of the likelihood must be calculated.

With the ARCH(1) model these are:

$$\frac{\partial l_t^b}{\partial \omega} = \frac{1}{2\sigma_t^2}\left(\frac{\varepsilon_t^2}{\sigma_t^2} - 1\right) \tag{13.10}$$

$$\frac{\partial l_t^b}{\partial \alpha} = \frac{1}{2\sigma_t^2}\varepsilon_{t-1}^2\left(\frac{\varepsilon_t^2}{\sigma_t^2} - 1\right) \tag{13.11}$$

$$\frac{\partial^2 l_t^b}{\partial \omega^2} = -\frac{1}{2\sigma_t^4}\left(2\frac{\varepsilon_t^2}{\sigma_t^2} - 1\right) \tag{13.12}$$

$$\frac{\partial^2 l_t^b}{\partial \omega^2} = -\frac{1}{2\sigma_t^4}\left(2\frac{\varepsilon_t^2}{\sigma_t^2} - 1\right) \tag{13.13}$$

$$\frac{\partial^2 l_t^b}{\partial \alpha^2} = -\frac{1}{2\sigma_t^4}\varepsilon_{t-1}^4\left(2\frac{\varepsilon_t^2}{\sigma_t^2} - 1\right) \tag{13.14}$$

$$\frac{\partial^2 l_t^b}{\partial \omega \partial \alpha} = -\frac{1}{2\sigma_t^4}\varepsilon_{t-1}^2\left(2\frac{\varepsilon_t^2}{\sigma_t^2} - 1\right). \tag{13.15}$$

The fist order conditions are $\sum_{t=2}^n \partial l_t^b/\partial\omega = 0$ and $\sum_{t=2}^n \partial l_t^b/\partial\alpha = 0$. The expected value of the second derivative has to be calculated for the same algorithm. It is assumed that $\mathsf{E}[Z_t^2] = \mathsf{E}[(\varepsilon_t/\sigma_t)^2] = 1$, so that the expression in the parentheses $(2\varepsilon_t^2/\sigma_t^2 - 1)$ has an expected value of one. From this it follows that:

$$\mathsf{E}\left[\frac{\partial^2 l_t^b}{\partial \omega^2}\right] = -\frac{1}{2}\,\mathsf{E}\left[\frac{1}{\sigma_t^4}\right].$$

The expectation of $\sigma_t^{-4}$ is consistently estimated by $(n-1)^{-1}\sum_{t=2}^n(\omega + \alpha\varepsilon_{t-1}^2)^{-2}$, so that for the estimator of the expected value of the second derivative we have:

$$\hat{\mathsf{E}}\frac{\partial^2 l_t^b}{\partial \omega^2} = -\frac{1}{2(n-1)}\sum_{t=2}^n \frac{1}{\sigma_t^4}.$$

Similarly the expected value of the second derivative with respect to $\alpha$ follows with:

$$\mathsf{E}\left[\frac{\partial^2 l_t^b}{\partial \alpha^2}\right] = -\frac{1}{2}\,\mathsf{E}\left[\frac{\varepsilon_{t-1}^4}{\sigma_t^4}\right]$$

and the estimator is:

$$\hat{\mathsf{E}}\frac{\partial^2 l_t^b}{\partial \alpha^2} = -\frac{1}{2(n-1)}\sum_{t=2}^n \frac{\varepsilon_{t-1}^4}{\sigma_t^4}.$$

**Theorem 13.6**
*Given $Z_t \sim N(0,1)$, it holds that:*

$$\mathsf{E}\left[\left(\frac{\partial l_t^b}{\partial \omega}\right)^2\right] = -\,\mathsf{E}\left[\frac{\partial^2 l_t^b}{\partial \omega^2}\right]$$

**Proof:**
This follows immediately from $\mathsf{E}\left[\left(\frac{\partial l_t^b}{\partial \omega}\right)^2\right] = \mathsf{E}\left[\frac{1}{4\sigma_t^4}(Z_t^4 - 2Z_t^2 + 1)\right]$ $= \mathsf{E}\left[\frac{1}{4\sigma_t^4}\right](3 - 2 + 1)$. □

Obviously Theorem 13.6 also holds for the parameter $\alpha$ in place of $\omega$. In addition it essentially holds for more general models, for example the estimation of GARCH models in Section 13.1.6. In more complicated models one can replace the second derivative with the square of the first derivative, which is easier to calculate. It is assumed, however, that the likelihood function is correctly specified, i.e., the true distribution of the error terms is normal.

Under the two conditions

1. $\mathsf{E}[Z_t \mid \mathcal{F}_{t-1}] = 0$ and $\mathsf{E}[Z_t^2 \mid \mathcal{F}_{t-1}] = 1$
2. $\mathsf{E}[\log(\alpha Z_t^2) \mid \mathcal{F}_{t-1}] < 0$ (strict stationarity)

and under certain technical conditions, the ML estimators are consistent. If $\mathsf{E}[Z_t^4 \mid \mathcal{F}_{t-1}] < \infty$ and $\omega > 0$, $\alpha > 0$ hold in addition, then $\hat{\theta} = (\hat{\omega}, \hat{\alpha})^\top$ is asymptotically normally distributed:

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{\mathcal{L}} \mathrm{N}(0, J^{-1} I J^{-1}) \tag{13.16}$$

with

$$I = \mathsf{E}\left(\frac{\partial l_t(\theta)}{\partial \theta} \frac{\partial l_t(\theta)}{\partial \theta^\top}\right)$$

and

$$J = -\mathsf{E}\left(\frac{\partial^2 l_t(\theta)}{\partial \theta \partial \theta^\top}\right).$$

If the true distribution of $Z_t$ is normal, then $I = J$ and the asymptotic covariance matrix is simplified to $J^{-1}$, i.e., the inverse of the Fischer Information matrix. If the true distribution is instead leptokurtic, then the maximum of (13.9) is still consistent, but no longer efficient. In this case the ML method is interpreted as the 'Quasi Maximum Likelihood' (QML) method.

In a Monte Carlo simulation study in Shephard (1996) 1000 ARCH(1) processes with $\omega = 0.2$ and $\alpha = 0.9$ were generated and the parameters were estimated using QML. The results are given in Table 13.2. Obviously with the moderate sample sizes ($n = 500$) the bias is negligible. The variance, however, is still so large that a relatively large proportion (10%) of the estimators are larger than one, which would imply covariance nonstationarity. This, in turn, has a considerable influence on the volatility prediction.

### 13.1.3 ARCH($q$): Definition and Properties

The definition of an ARCH(1) model will be extended for the case that $q > 1$ lags, on which the conditional variance depends.

| $n$ | $k^{-1}\sum_{j=1}^{k}\hat{\alpha}_j$ | $\sqrt{k^{-1}\sum_{j=1}^{k}(\hat{\alpha}_j-\alpha)^2}$ | $\#(\alpha_j \geq 1)$ |
|---|---|---|---|
| 100 | 0.852 | 0.257 | 27% |
| 250 | 0.884 | 0.164 | 24% |
| 500 | 0.893 | 0.107 | 15% |
| 1000 | 0.898 | 0.081 | 10% |

Table 13.2: Monte Carlo simulation results for QML estimates of the parameter $\alpha = 0.9$ from an ARCH(1) model with $k = 1000$ replications. The last column gives the proportion of the estimator that are larger than 1 (according to Shephard (1996)).

**Definition 13.2 (ARCH($q$))**
*The process* $(\varepsilon_t)$, $t \in \mathbb{Z}$, *is ARCH(q), when* $\mathsf{E}[\varepsilon_t \mid \mathcal{F}_{t-1}] = 0$,

$$\sigma_t^2 = \omega + \alpha_1\varepsilon_{t-1}^2 + \ldots + \alpha_q\varepsilon_{t-q}^2 \tag{13.17}$$

*with* $\omega > 0$, $\alpha_1 \geq 0, \ldots, \alpha_q \geq 0$ *and*

- $Var(\varepsilon_t \mid \mathcal{F}_{t-1}) = \sigma_t^2$ *and* $Z_t = \varepsilon_t/\sigma_t$ *is i.i.d. (strong ARCH)*
- $Var(\varepsilon_t \mid \mathcal{F}_{t-1}) = \sigma_t^2$ *(semi-strong ARCH), or*
- $\mathcal{P}(\varepsilon_t^2 \mid 1, \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots, \varepsilon_{t-1}^2, \varepsilon_{t-2}^2, \ldots) = \sigma_t^2$ *(weak ARCH)*

The conditional variance $\sigma_t^2$ in an ARCH($q$) model is also a linear function of the $q$ squared lags.

**Theorem 13.7**
*Let* $\varepsilon_t$ *be a semi-strong ARCH(q) process with* $Var(\varepsilon_t) = \sigma^2 < \infty$. *Then*

$$\sigma^2 = \frac{\omega}{1 - \alpha_1 - \ldots - \alpha_q}$$

*with* $\alpha_1 + \cdots + \alpha_q < 1$.

**Proof:**
as in Theorem 13.2. □

If instead $\alpha_1 + \cdots + \alpha_q \geq 1$, then the unconditional variance does not exist and the process is not covariance-stationary.

**Theorem 13.8 (Representation of an ARCH($q$) Process)**
*Let* $\varepsilon_t$ *be a (semi-)strong ARCH(q) process with* $\mathsf{E}[\varepsilon_t^4] = c < \infty$. *Then*

1. $\eta_t = \sigma_t^2(Z_t^2 - 1)$ *is white noise.*
2. $\varepsilon_t^2$ *is an AR(q) process with* $\varepsilon_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \eta_t$.

**Proof:**
as in Theorem 13.5. □

It is problematic with the ARCH($q$) model that for some applications a larger order $q$ must be used, since large lags only lose their influence on the volatility slowly. It is suggested as an empirical rule of thumb, that a minimum order of $q = 14$ should be used. The disadvantage of a large order is that many parameters have to be estimated under restrictions. The restrictions can be categorised as conditions for stationarity and the strictly positive parameters. If efficient estimation methods are to be used, for example, the maximum likelihood method, the estimation of large dimensional parameter spaces can be numerically quite complicated to obtain.

One possibility of reducing the number of parameters while including a long history is to assume linearly decreasing weights on the lags, i.e.,

$$\sigma_t^2 = \omega + \alpha \sum_{i=1}^{q} w_i \varepsilon_{t-i}^2,$$

with

$$w_i = \frac{2(q+1-i)}{q(q+1)},$$

so that only two parameters need to be estimated. In Section 13.1.5 we describe a generalised ARCH model, which on the one hand, has a parsimonious parameterization, and on the other hand a flexible lag structure.

### 13.1.4 Estimation of an ARCH($q$) Model

For the general ARCH($q$) model from (13.17) the conditional likelihood is

$$\begin{aligned} l^b(\theta) &= \sum_{t=q+1}^{n} l_t(\theta) \\ &= -\frac{n-1}{2}\log(2\pi) - 1/2 \sum_{t=2}^{n} \log \sigma_t^2 - 1/2 \sum_{t=q+1}^{n} \frac{\varepsilon_t^q + 1}{\sigma_t^2} \end{aligned} \quad (13.18)$$

with the parameter vector $\theta = (\omega, \alpha_1, \ldots, \alpha_q)^\top$. Although one can find the optimum of ARCH(1) models by analysing the graph such as Figure 13.5, it

is complicated and impractical for a high dimensional parameter space. The maximisation of (13.18) with respect to $\theta$ is a non-linear optimisation problem, which can be solved numerically. The *score algorithm* is used empirically not only in ARMA models (see Section 12.8) but also in ARCH models. In order to implement this approach the first and second derivatives of the (conditional) likelihood with respect to the parameters need to be formed. For the ARCH($q$) model the first derivative is:

$$\frac{\partial l_t^b}{\partial \theta} = \frac{1}{2\sigma_t^2} \frac{\partial \sigma_t^2}{\partial \theta} \left( \frac{\varepsilon_t^2}{\sigma_t^2} - 1 \right) \tag{13.19}$$

with

$$\frac{\partial \sigma_t^2}{\partial \theta} = (1, \varepsilon_{t-1}^2, \ldots, \varepsilon_{t-q}^2)^\top.$$

The first order condition is $\sum_{t=q+1}^{n} \partial l_t / \partial \theta = 0$. For the second derivative and the asymptotic properties of the QML estimator see Section 13.1.6.

### 13.1.5 Generalised ARCH (GARCH)

The ARCH($q$) model can be generalised by extending it with autoregressive terms of the volatility.

**Definition 13.3 (GARCH($p, q$))** *The process* $(\varepsilon_t)$, $t \in \mathbb{Z}$, *is GARCH($p, q$), if* $\mathsf{E}[\varepsilon_t \mid \mathcal{F}_{t-1}] = 0$,

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2, \tag{13.20}$$

*and*

- $Var(\varepsilon_t \mid \mathcal{F}_{t-1}) = \sigma_t^2$ *and* $Z_t = \varepsilon_t / \sigma_t$ *is i.i.d. (strong GARCH)*
- $Var(\varepsilon_t \mid \mathcal{F}_{t-1}) = \sigma_t^2$ *(semi-strong GARCH), or*
- $\mathcal{P}(\varepsilon_t^2 \mid 1, \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots, \varepsilon_{t-1}^2, \varepsilon_{t-2}^2, \ldots) = \sigma_t^2$ *(weak GARCH).*

The sufficient but not necessary conditions for

$$\sigma_t^2 > 0 \quad a.s., \qquad (\ \mathrm{P}[\sigma_t^2 > 0] = 1\ ) \tag{13.21}$$

are $\omega > 0$, $\alpha_i \geq 0$, $i = 1, \ldots, q$ and $\beta_j \geq 0$, $j = 1, \ldots, p$. In the case of the GARCH(1,2) model

$$\begin{aligned}\sigma_t^2 &= \omega + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \beta_1 \sigma_{t-1}^2 \\ &= \frac{\omega}{1-\beta} + \alpha_1 \sum_{j=0}^{\infty} \beta_1^j \varepsilon_{t-j-1}^2 + \alpha_2 \sum_{j=0}^{\infty} \beta_1^j \varepsilon_{t-j-2}^2 \\ &= \frac{\omega}{1-\beta} + \alpha_1 \varepsilon_{t-1}^2 + (\alpha_1 \beta_1 + \alpha_2) \sum_{j=0}^{\infty} \beta_1^j \varepsilon_{t-j-2}^2\end{aligned}$$

with $0 \leq \beta_1 < 1$. $\omega > 0$, $\alpha_1 \geq 0$ and $\alpha_1 \beta_1 + \alpha_2 \geq 0$ are necessary and sufficient conditions for (13.21) assuming that the sum $\sum_{j=0}^{\infty} \beta_1^j \varepsilon_{t-j-2}^2$ converges.

**Theorem 13.9 (Representation of a GARCH($p, q$) process)**
*Let $\varepsilon_t$ be a (semi-)strong GARCH($p, q$) process with $\mathsf{E}[\varepsilon_t^4] = c < \infty$. Then*

1. *$\eta_t = \sigma_t^2 (Z_t^2 - 1)$ is white noise.*
2. *$\varepsilon_t^2$ is an ARMA($m, p$) process with*

$$\varepsilon_t^2 = \omega + \sum_{i=1}^{m} \gamma_i \varepsilon_{t-i}^2 - \sum_{j=1}^{p} \beta_j \eta_{t-j} + \eta_t, \tag{13.22}$$

*with $m = \max(p, q)$, $\gamma_i = \alpha_i + \beta_i$. $\alpha_i = 0$ when $i > q$, and $\beta_i = 0$ when $i > p$.*

**Proof:**
as in Theorem 13.5. □

If $\varepsilon_t$ follows a GARCH process, then from Theorem 13.9 we can see that $\varepsilon_t^2$ follows an ARMA model with conditional heteroscedastic error terms $\eta_t$. As we know, if all the roots of the polynomial $(1 - \beta_1 z - \ldots - \beta_p z^p)$ lie outside the unit circle, then the ARMA process (13.22) is invertible and can be written as an AR($\infty$) process. Moveover it follows from Theorem 13.8 that the GARCH($p, q$) model can be represented as an ARCH($\infty$) model. Thus one can deduce analogous conclusions from the ARMA models in determining the order $(p, q)$ of the model. There are however essential differences in the definition of the persistence of shocks.

**Theorem 13.10 (Unconditional variance of a GARCH$(p,q)$ process)**

*Let $\varepsilon_t$ be a semi-strong GARCH$(p,q)$ process with $Var(\varepsilon_t) = \sigma^2 < \infty$. Then*

$$\sigma^2 = \frac{\omega}{1 - \sum_{i=1}^{q} \alpha_i - \sum_{j=1}^{p} \beta_j},$$

*with $\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1$.*

**Proof:**
as in Theorem 13.2. □

General conditions for the existence of higher moments of the GARCH$(p,q)$ models are given in He and Teräsvirta (1999). For the smaller order models and under the assumption of distribution we can derive:

**Theorem 13.11 (Fourth moment of a GARCH(1,1) process)**
*Let $\varepsilon_t$ be a (semi-)strong GARCH(1,1) process with $Var(\varepsilon_t) = \sigma^2 < \infty$ and $Z_t \sim N(0,1)$. Then $\mathsf{E}[\varepsilon_t^4] < \infty$ holds if and only if $3\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2 < 1$. The Kurtosis $Kurt(\varepsilon_t)$ is given as*

$$Kurt[\varepsilon_t] = \frac{\mathsf{E}[\varepsilon_t^4]}{\left(\mathsf{E}[\varepsilon_t^2]\right)^2} = 3 + \frac{6\alpha_1^2}{1 - \beta_1^2 - 2\alpha_1\beta_1 - 3\alpha_1^2}. \tag{13.23}$$

**Proof:**
It can be proved that $\mathsf{E}[\varepsilon_t^4] = 3\,\mathsf{E}[(\omega + \alpha_1\varepsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2)^2]$ and the stationarity of $\varepsilon_t$. □

The function (13.23) is illustrated in Figure 13.6 for all $\alpha_1 > 0$, $\text{Kurt}[\varepsilon_t] > 3$, i.e., the distribution of $\varepsilon_t$ is leptokurtic. We can observe that the kurtosis equals 3 only in the case of the boundary value $\alpha_1 = 0$ where the conditional heteroscedasticity disappears and a Gaussian white noise takes place. In addition it can be seen in the figure that the kurtosis increases in $\beta_1$ slowly for a given $\alpha_1$. On the contrary it increases in $\alpha_1$ much faster for a given $\beta_1$.

**Remark 13.3**
*Nelson (1990a) shows that the strong GARCH(1,1) process $X_t$ is strictly stationary when $\mathsf{E}[\log(\alpha_1 Z_t^2 + \beta_1)] < 0$. If $Z_t \sim N(0,1)$, then the conditions for strict stationarity are weaker than those for covariance-stationarity: $\alpha_1 + \beta_1 < 1$.*

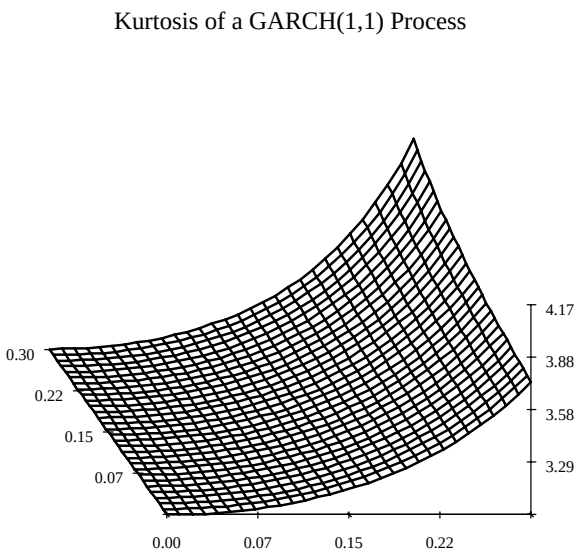


Figure 13.6: Kurtosis of a GARCH(1,1) process according to (13.23). The left axis shows the parameter $\beta_1$, the right $\alpha_1$. SFEkurgarch

In practical applications it is frequently shown that models with smaller order sufficiently describe the data. In most cases GARCH(1,1) is sufficient.

A substantial disadvantage of the standard ARCH and GARCH models exists since they cannot model asymmetries of the volatility with respect to the sign of past shocks. This results from the squared form of the lagged shocks in (13.17) and (13.20). Therefore they have an effect on the level but no effect on the sign. In other words, bad news (identified by a negative sign) has the same influence on the volatility as good news (positive sign) if the absolute values are the same. Empirically it is observed that bad news has a larger effect on the volatility than good news. In Section 13.2 and 14.1 we will take a closer look at the extensions of the standard models which can be used to calculate these observations.

### 13.1.6 Estimation of GARCH($p, q$) Models

Based on the ARMA representation of GARCH processes (see Theorem 13.9) Yule-Walker estimators $\tilde{\theta}$ are considered once again. These estimators

are, as can be shown, consistent and asymptotically normally distributed, $\sqrt{n}(\tilde{\theta} - \theta) \xrightarrow{\mathcal{L}} \mathrm{N}(0, \tilde{\Sigma})$. However in the case of GARCH models they are not efficient in the sense that the matrix $\tilde{\Sigma} - J^{-1}IJ^{-1}$ is positively definite, where $J^{-1}IJ^{-1}$ is the asymptotic covariance matrix of the QML estimator, see (13.26). In the literature there are several experiments on the efficiency of the Yule-Walker and QML estimators in finite samples, see Section 13.5. In most cases maximum likelihood methods are chosen in order to get the efficiency.

The likelihood function of the general GARCH$(p, q)$ model (13.20) is identical to (13.18) with the extended parameter vector $\theta = (\omega, \alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_p)^\top$. Figure 13.7 displays the likelihood function of a generated GARCH(1,1) process with $\omega = 0.1$, $\alpha = 0.1$, $\beta = 0.8$ and $n = 500$. The parameter $\omega$ was chosen so that the unconditional variance is constant everywhere, i.e., with a variance of $\sigma^2$, $\omega = (1 - \alpha - \beta)\sigma^2$. As one can see, the function is flat on the right, close to the optimum, thus the estimation will be relatively imprecise, i.e., it will have a larger variance. In addition, Figure 13.8 displays the contour plot of the likelihood function.

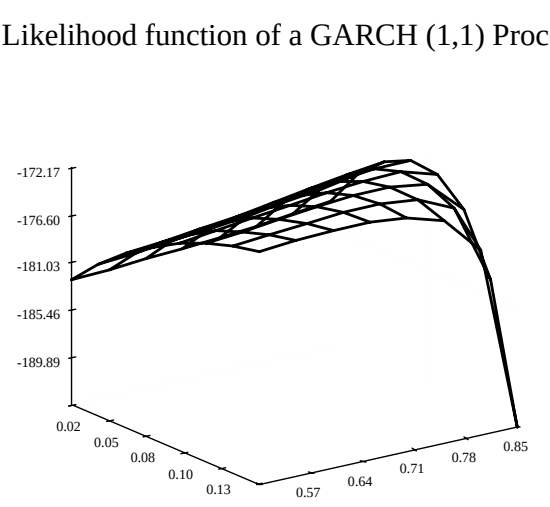


Figure 13.7: Likelihood function of a generated GARCH(1,1) process with $n = 500$. The left axis shows the parameter $\beta$, the right $\alpha$. The true parameters are $\omega = 0.1$, $\alpha = 0.1$ and $\beta = 0.8$. SFElikgarch

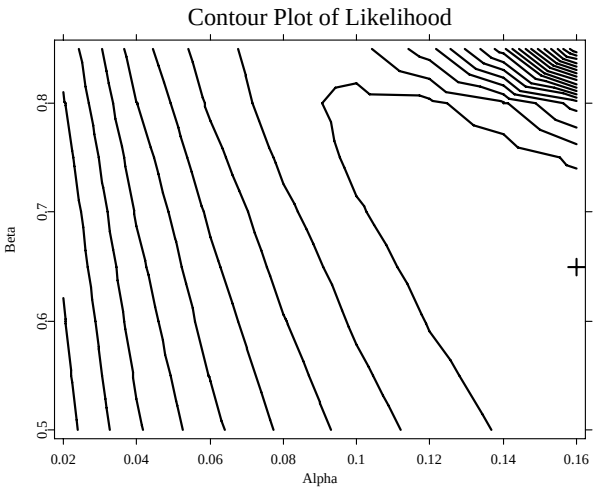


Figure 13.8: Contour plot of the likelihood function of a generated GARCH(1,1) process with $n = 500$. The perpendicular axis displays the parameter $\beta$, the horizontal $\alpha$. The true parameters are $\omega = 0.1$, $\alpha = 0.1$ and $\beta = 0.8$. SFElikgarch

The first partial derivatives of (13.18) are

$$\frac{\partial l_t}{\partial \theta} = \frac{1}{2\sigma_t^2} \frac{\partial \sigma_t^2}{\partial \theta} \left( \frac{\varepsilon_t^2}{\sigma_t^2} - 1 \right) \tag{13.24}$$

with

$$\frac{\partial \sigma_t^2}{\partial \theta} = \vartheta_t + \sum_{j=1}^{p} \frac{\partial \sigma_{t-j}^2}{\partial \theta}.$$

and $\vartheta_t = (1, \varepsilon_{t-1}^2, \ldots, \varepsilon_{t-q}^2, \sigma_{t-1}^2, \ldots, \sigma_{t-p}^2)^\top$. The first order conditions are $\sum_{t=q+1}^{n} \partial l_t / \partial \theta = 0$. The matrix of the second derivatives takes the following form:

$$\begin{aligned} \frac{\partial^2 l_t(\theta)}{\partial \theta \partial \theta^\top} &= \frac{1}{2\sigma_t^4} \frac{\partial \sigma_t^2}{\partial \theta} \frac{\partial \sigma_t^2}{\partial \theta^\top} - \frac{1}{2\sigma_t^2} \frac{\partial^2 \sigma_t^2(\theta)}{\partial \theta \partial \theta^\top} \\ &- \frac{\varepsilon_t^2}{\sigma_t^6} \frac{\partial \sigma_t^2}{\partial \theta} \frac{\partial \sigma_t^2}{\partial \theta^\top} + \frac{\varepsilon_t^2}{2\sigma_t^4} \frac{\partial^2 \sigma_t^2(\theta)}{\partial \theta \partial \theta^\top} \end{aligned} \tag{13.25}$$

Under the conditions

1. $\mathsf{E}[Z_t \mid \mathcal{F}_{t-1}] = 0$ and $\mathsf{E}[Z_t^2 \mid \mathcal{F}_{t-1}] = 1$,
2. strict stationarity of $\varepsilon_t$

and under some technical conditions the ML estimator is consistent. If in addition it holds that $\mathsf{E}[Z_t^4 \mid \mathcal{F}_{t-1}] < \infty$, then $\hat{\theta}$ is asymptotically normally distributed:

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{\mathcal{L}} \mathrm{N}_{p+q+1}(0, J^{-1} I J^{-1}) \tag{13.26}$$

with

$$I = \mathsf{E}\left(\frac{\partial l_t(\theta)}{\partial \theta} \frac{\partial l_t(\theta)}{\partial \theta^\top}\right)$$

and

$$J = -\,\mathsf{E}\left(\frac{\partial^2 l_t(\theta)}{\partial \theta \partial \theta^T}\right).$$

**Theorem 13.12 (Equivalence of $I$ and $J$)**
*If $Z_t \sim N(0,1)$, then it holds that $I = J$.*

**Proof:**
Building the expectations of (13.25) one obtains

$$\mathsf{E}\left[\frac{\partial^2 l_t(\theta)}{\partial \theta \partial \theta^\top}\right] = -\,\mathsf{E}\,\frac{1}{2\sigma_t^4} \frac{\partial \sigma_t^2}{\partial \theta} \frac{\partial \sigma_t^2}{\partial \theta^\top}.$$

For $I$ we have

$$\begin{aligned} \mathsf{E}\left[\frac{\partial l_t(\theta)}{\partial \theta} \frac{\partial l_t(\theta)}{\partial \theta^T}\right] &= \mathsf{E}\left[\frac{1}{4\sigma_t^4} \frac{\partial \sigma_t^2}{\partial \theta} \frac{\partial \sigma_t^2}{\partial \theta^T} (Z_t^4 - 2Z_t^2 + 1)\right] \\ &= \mathsf{E}\left[\frac{1}{4\sigma_t^4} \frac{\partial \sigma_t^2}{\partial \theta} \frac{\partial \sigma_t^2}{\partial \theta^\top}\right] \{\mathrm{Kurt}(Z_t \mid \mathcal{F}_{t-1}) - 1\} \end{aligned} \tag{13.27}$$

From the assumption $Z_t \sim \mathrm{N}(0,1)$ it follows that $\mathrm{Kurt}(Z_t \mid \mathcal{F}_{t-1}) = 3$ and thus the claim. □

If the distribution of $Z_t$ is specified correctly, then $I = J$ and the asymptotic variance can be simplified to $J^{-1}$, i.e., the inverse of the Fisher Information matrix. If this is not the case and it is instead leptokurtic, for example, the maximum of (13.9) is still consistent but no longer efficient. In this case the ML method is interpreted as the 'Quasi Maximum Likelihood' (QML) method.

Consistent estimators for the matrices $I$ and $J$ can be obtained by replacing the expectation with the simple average.

## 13.2 Extensions of the GARCH Model

Standard GARCH models assume that positive and negative error terms have a symmetric effect on volatility. In other words, good and bad news have the same effect on the volatility in this model. In practice this assumption is frequently violated, in particular by stock returns, in that the volatility increases more after bad news than after good news. This so called *Leverage Effect* appears firstly in Black (1976), who noted that:

> *"a drop in the value of the firm will cause a negative return on its stock, and will usually increase the leverage of the stock. [...] That rise in the debt-equity ratio will surely mean a rise in the volatility of the stock".*

A very simple but plausible explanation for the leverage effect: Negative returns imply a larger proportion of debt through a reduced market value of the firm, which leads to a higher volatility. The risk, i.e. the volatility reacts first to larger changes of the market value, nevertheless it is empirically shown that there is a high volatility after smaller changes. On the other hand, Black said nothing about the effect of positive returns on the volatility. Although the positive returns cause smaller increases, they do cause an increase in the volatility. From an empirical point of view the volatility reacts asymmetrically to the sign of the shocks and therefore a number of parameterized extensions of the standard GARCH model have been suggested recently. Below we will discuss two of the most important ones: the exponential GARCH (EGARCH) and the threshold GARCH (TGARCH) model.

### 13.2.1 Exponential GARCH

Let $Z_t$ further denote a series of *i.i.d.* standardised random variables with expectation 0 and variance 1. The general exponential GARCH (EGARCH) model is given by Nelson (1991):

$$\log \sigma_t^2 = \omega_t + \sum_{k=1}^{\infty} \beta_k g(Z_{t-k}), \tag{13.28}$$

where $\omega_t$, $\beta_k$ are deterministic coefficients and

$$g(Z_t) = \theta Z_t + \gamma(|Z_t| - E|Z_t|). \tag{13.29}$$

It can be directly seen that $E\left[g(Z_t)\right] = 0$.

The EGARCH model in (13.28) shows some differences from the standard GARCH model:

- Volatility of the EGARCH model, which is measured by the conditional variance $\sigma_t^2$, is an explicit multiplicative function of lagged innovations. On the contrary, volatility of the standard GARCH model is an additive function of the lagged error terms $\varepsilon_t$, which causes a complicated functional dependency on the innovations; for example the ARCH(1) in Theorem 13.4.
- Volatility can react asymmetrically to the good and the bad news.
- For the general distributions of $Z_t$ the parameter restrictions for strong and covariance-stationarity coincide.
- The parameters in (13.28) and (13.29) are not restricted to positive values.

The function $g(\cdot)$ in (13.29) is piecewise linear. It contains two parameters which define the 'size effect' and the 'sign effect' of the shocks on volatility. The first is a typical ARCH effect while the second is an asymmetrical effect, for example, the leverage effect. The term $\gamma(|Z_t| - E|Z_t|)$ determines the size effect and the term $\theta Z_t$ determines the sign effect. The parameter $\gamma$ is thus typically positive and $\theta$ is negative.

To estimate EGARCH models instead of the general MA($\infty$) representation (13.28), an ARMA($p, q$) model is applied, i.e.,

$$\Delta(L) \log \sigma_t^2 = \omega + \Psi(L) g(Z_t), \tag{13.30}$$

with lag-polynomial $\Delta(L)$ and $\Psi(L)$ of order $p$ and $q$ respectively.

EGARCH models benefit from no parameter restrictions, thus the possible instabilities of optimisation routines are reduced. On the other hand the theoretical properties of QML estimators of EGARCH models are not clarified to a great extent.

Let $\omega_t = \omega = 0$ and $\sum_{k=1}^{\infty} \beta_k^2 < \infty$. Then $\sigma_t^2$ is strictly stationary and ergodic, see Theorem 2.1 in Nelson (1991). Furthermore, under these conditions the unconditional variance exists when $Z_t$ has a *generalised error distribution* (GED) with parameter $\zeta > 1$, which determines the thickness of the tails, see Theorem 2.2 in Nelson (1991). The GED is leptokurtic when $\zeta < 2$.

The normal distribution is a special case of the GED ($\zeta = 2$). In addition, Nelson gives complicated formulas for the unconditional moments. One problem is that under other leptokurtic distributions such as the Student-$t$, the unconditional variance does not exist. The reason is that exponential growth of the conditional variance changes with the level of the shocks, which leads to the explosion of the unconditional variance when the probability for extreme shocks is sufficiently large. Therefore the existence of the unconditional

moments depends on the choice of the distribution of the innovations, which is an undesirable property of the EGARCH models. In empirical studies it has been found that EGARCH often overweighs the effects of larger shocks on volatility and thus results in poorer fits than standard GARCH models, see the empirical studies of Engle and Ng (1993).

### 13.2.2 Threshold ARCH Models

The idea of the Threshold ARCH (TARCH) models is to divide the distribution of the innovations into disjointed intervals and then approximate a piecewise linear function for the conditional standard deviation, see Zakoian (1991), and the conditional variance respectively, see Glosten, Jagannathan and Runkle (1993). If there are only two intervals, the division is normally at zero, i.e., the influence of positive and negative innovations on the volatility is differentiated. In this case the TARCH model of order $q$ can be written as:

$$\sigma_t^\delta = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^\delta + \sum_{i=1}^{q} \alpha_i^- \varepsilon_{t-i}^\delta I(\varepsilon_{t-i} < 0), \tag{13.31}$$

with the indicator function $I(\cdot)$ and $\delta = 1$ as in Zakoian (1991) or $\delta = 2$ as in Glosten et al. (1993).

Rabemananjara and Zakoian (1993) extend this model by including the lagged conditional standard deviations (variance respectively) as a regressor, which is known as the TGARCH model. They also give conditions for covariance-stationarity in their study.

Instead of a piecewise linear function Gouriéroux and Monfort (1992) use a stepwise function (piecewise constant) as a model for the volatility. Let $A_j, j = 1, \ldots, J$ be a partition of the distribution of the innovation. Then a *qualitative threshold ARCH* model (QTARCH) of order 1 is given by:

$$y_t = \sum_{j=1}^{J} m_j I(y_{t-1} \in A_j) + \sum_{j=1}^{J} s_j I(y_{t-1} \in A_j) Z_t, \tag{13.32}$$

where $m_j$ and $s_j$ are scalars. In (13.32) conditional expectations and conditional standard deviations are modelled as stepwise functions. One notices that (13.32) is a homogenous Markov Chain of order one. Models of higher order can be easily derived. Gouriéroux and Monfort (1992) give a detailed discussion of the statistical properties of (13.32). Moreover the consistency and asymptotic normal distribution of the QML estimators are also discussed by them.

The threshold models are identified by an abrupt transition between two regimes when the generating innovation crosses a threshold value. If a smooth transition is preferred in the model, then the indicator function in (13.31) can be replaced with the desired continuous function, which tends to zero if the values are close to $\varepsilon_{t-i}$ and tends to one for the values further away. Frequently, the logistic function is chosen. The model is thus:

$$\sigma_t^\delta = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^\delta + \sum_{i=1}^{q} \alpha_i^- \varepsilon_{t-i}^\delta F_\gamma(\varepsilon_{t-i}) \tag{13.33}$$

$$F_\gamma(u) = \frac{1}{1+\exp(-\gamma u)}, \tag{13.34}$$

with the parameter $\gamma > 0$. The indicator function is a limiting case of the logistic function for $\gamma \longrightarrow \infty$.

Finally, another model class needs to be mentioned, which is very general and can replicate the asymmetries: the Asymmetric Power ARCH (APARCH) model from Ding, Granger and Engle (1993),

$$\sigma_t^\delta = \omega + \sum_{i=1}^{q} \alpha_i (|\varepsilon_{t-i}| - \gamma_i \varepsilon_{t-i})^\delta + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^\delta, \tag{13.35}$$

where $\delta > 0$ is a parameter to be estimated. However, note that the EGARCH model is not included in this model class, a direct test between GARCH and EGARCH models is thus impossible. A very general ARCH model, the *augmented GARCH* model from Duan (1997), also includes the EGARCH model.

### 13.2.3 Risk and Returns

In finance theory the relationship between risk and returns plays an important role. Many theoretical models such as CAPM, imply a linear relationship between the expected returns of a market portfolio and the variance. If the risk (i.e. the variance) is not constant over time, then the conditional expectation of the market returns is a linear function of the conditional variance. The idea from Engle, Lilien and Robins (1987) was consequently used to estimate the conditional variances in GARCH and then the estimations used in the conditional expectations' estimation. This is the so called *ARCH in Mean* (ARCH–M) model.

Let $y_t$ be a covariance-stationary return process of a broad market index and $\sigma_t^2$ be the conditional variance specified in an ARCH(1) model. The

ARCH-M model is

$$\begin{aligned} y_t &= \nu + \lambda g(\sigma_t^2) + \varepsilon_t \\ \sigma_t^2 &= \omega + \alpha \varepsilon_{t-1}^2, \end{aligned} \tag{13.36}$$

where $g(\cdot)$ is a known parameter function. The CAPM implies that $g(x) = x$, although in most empirical applications the square root or the logarithm is used. The expression $\lambda g(\sigma_t^2)$ can be interpreted as the risk premium.

### 13.2.4 Estimation Results for the DAX Returns

We applied various ARCH models discussed previously to the returns of the German stock index (DAX). Here we didn't use the DAX as quoted on the stock exchange, but instead the DAFOX from the capital market database in Karlsruhe, which was created for analytical purposes and is adjusted for dividends and inflation. We consider daily returns from 1974 to 1996 (5748 observations).

The returns indicate a clear auto-correlation of first order. There are various possibilities to model these auto-correlations. The two most important models, which we have discussed, are the AR models and the ARCH-M models. The latter is easy to interpret economically, i.e., a time dependent risk premium implying an auto-correlation of the returns. This economic motivation is missing for the AR models: the AR term cannot be interpreted as the risk premium, since it can be negative, which contradicts the usual assumption of a risk averse agent. However the AR models frequently offer a better fit to the data than the ARCH-M model. The basic model is thus:

$$y_t = \mu_t + \sigma_t Z_t$$

with $\mu_t = \nu + \phi y_{t-1}$ (AR(1)) respectively $\mu_t = \nu + \lambda \sigma_t$ (ARCH-M), and $\sigma_t$ can be GARCH, TGARCH or EGARCH.

The estimation results are given in Table 13.3. They can be summarised as follows:

- The parameter $\nu$ is significantly positive for all AR models, which is reflected in the long-run increasing trend of the stock price. For the ARCH-M model the sign of the trends is not only given in $\nu$ but also in $\lambda$. The effect of a negative $\nu$ can be dominated by a positive $\lambda$, which is the case in the GARCH-M and the EGARCH-M models.
- The ARCH effects are very pronounced, i.e., the parameter $\alpha$ in the GARCH model is significant.

| | AR-G | AR-TG | AR-EG | GARCH-M | TGARCH-M | EGARCH-M |
|---|---|---|---|---|---|---|
| $\lambda$ | | | | 0.107 | 0.051 | 0.338 |
| | | | | (2.14) | (1.09) | (3.31) |
| $\nu$ | 3.75E-04 | 3.03E-04 | 2.57E-04 | -2.84E-04 | 4.71E-05 | -0.002 |
| | (4.414) | (3.290) | (2.608) | (-0.79) | (0.14) | (-2.72) |
| $\phi$ | 0.131 | 0.137 | 0.136 | | | |
| | (6.808) | (8.079) | (5.940) | | | |
| $\omega$ | 8.42E-07 | 8.22E-07 | -5.259 | 8.81E-07 | 8.78E-07 | -3.086 |
| | (5.312) | (5.444) | (-3.692) | (5.28) | (5.45) | (-4.79) |
| $\alpha$ | 0.079 | 0.045 | | 0.076 | 0.049 | |
| | (3.004) | (2.073) | | (2.74) | (1.94) | |
| $\alpha^-$ | | 0.058 | | | 0.044 | |
| | | (1.752) | | | (1.77) | |
| $\beta$ | 0.914 | 0.918 | 0.490 | 0.917 | 0.919 | 0.711 |
| | (44.07) | (38.18) | (3.41) | (41.43) | (36.57) | (10.87) |
| $\gamma$ | | | 0.431 | | | 0.388 |
| | | | (13.95) | | | (11.25) |
| $\theta$ | | | -0.171 | | | -0.085 |
| | | | (-3.65) | | | (-1.97) |
| logL | 20030.86 | 20049.18 | 19713.81 | 19996.19 | 20008.22 | 19827.41 |

Table 13.3: Estimation results of various ARCH models, applied to DAFOX returns 1974–1996. Parenthesis show the $t$-statistic based on the QML asymptotic standard error.

- There is a high persistence of shocks in the volatility. This persistence is measured in the GARCH case by the sum of $\alpha$ and $\beta$ and is in each case close to 1.
- Except for the EGARCH specification of the volatility the AR(1) model describes the data better than the ARCH-M models.
- A leverage effect exists: the corresponding parameters in the TGARCH and the EGARCH models have the appropriate signs. In the TGARCH case the $t$-statistic for $\alpha^-$ is 1.75 and 1.77 respectively, and in the EGARCH case the $t$-statistic for $\theta$ is -3.65 and -1.97 respectively. Negative shocks increase the volatility more than positive shocks.
- The TGARCH and the EGARCH models have a priority for asymmetry, since they have a better fit to the data when the same number of parameters are considered.

## 13.3 Shortfalls of GARCH

The purpose of this section is to examine some shortfalls uncovered in previous academic literature, primarily GARCH's inability to function as a true data generating process and accurate forecasting tool. Furthermore, this section provides an explanation of these problems which include GARCH's overestimation of the unconditional variance and the existence of the IGARCH effect. Section 13.3.1 discusses analysis of returns of the S&P 500 stock market index from Starica (2003). Section 13.3.2 presents original analysis of DAX returns for comparison.

### 13.3.1 Recent Challenges to GARCH Models

One of the main sources criticising the ability of GARCH models to accurately model financial time series is the paper by Starica (2003) which discusses how well the endogenous dynamics created by the GARCH (1,1) model replicate the actual dynamics of main financial indices. Specifically, Starica tests the GARCH model's performance in describing and forecasting the dynamics of returns of the S&P 500 stock market index.

#### Modelling S&P 500 Data

Fitting a conditionally normal GARCH (1,1) process $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$ to the S&P 500 daily logarithmic returns (first differences of the logarithms of daily closing prices) from 16 May, 1995 to 29 April, 2003, a range containing exactly 2000 observations, yields the estimated parameters:

$$\widehat{\omega} = 1.4264 \times 10^{-6}, \qquad \widehat{\alpha} = 0.0897, \qquad \widehat{\beta} = 0.9061. \tag{13.37}$$

As $\alpha + \beta = 0.995$, a value very close to 1, the integrated GARCH (IGARCH) effect is present.

**IGARCH Effect**

When estimating the parameters in GARCH models, it is observed that for shorter samples, the estimated parameters $\alpha$ and $\beta$ sum up to values significantly different from 1 while for longer samples, their sum approaches 1. The combination of these phenomenon is called the integrated GARCH (IGARCH) effect of return data.

Stationarity of the data is one of the basic assumptions of GARCH models. The IGARCH effect occurs due to non-stationarity of the data as a result of

structural breaks, i.e. changes in the unconditional mean or variance. For a GARCH (1,1) process, when $\alpha + \beta$ approaches 1, this signals a presence of an IGARCH effect. The larger the absolute difference in the variance (the greater the degree of non-stationarity) of sub-samples having differing unconditional variances, the closer $\alpha + \beta$ gets to 1.

Structural breaks in unconditional variance resulting in IGARCH effects pose problems when using GARCH models to estimate volatilities. IGARCH is likely to cause an explosion of the estimated unconditional variance of the GARCH processes.

### Next-day Volatility Forecasting

In practice, one of GARCH's main applications is to provide next-day volatility forecasts. When estimated innovations, or residuals, defined as

$$\widehat{Z}_t = \varepsilon_t/\widehat{\sigma}_t, \qquad \widehat{\sigma}_t^2 = \widehat{\omega} + \widehat{\alpha}\varepsilon_{t-1}^2 + \widehat{\beta}\widehat{\sigma}_{t-1}^2, \qquad t = 1, ....n \tag{13.38}$$

are almost independent, this is taken as evidence of accurate next-day volatility forecasts.

GARCH (1,1) volatility can be found using previously estimated parameters applied to the GARCH (1,1) model. However, for the S&P 500 data, the following infinite moving-average approximation also provides an accurate estimate.

$$\begin{aligned}
\widehat{\sigma}_t^2 &= \widehat{\omega} + \widehat{\alpha}\varepsilon_{t-1}^2 + \widehat{\beta}\widehat{\sigma}_{t-1}^2 \\
&= \widehat{\omega} + \widehat{\alpha}\varepsilon_{t-1}^2 + \widehat{\beta}(\widehat{\omega} + \widehat{\alpha}\varepsilon_{t-2}^2 + \widehat{\beta}\widehat{\sigma}_{t-2}^2) \\
&= \widehat{\omega}(1+\widehat{\beta}) + \widehat{\alpha}(\varepsilon_{t-1}^2 + \widehat{\beta}\varepsilon_{t-2}^2) + \widehat{\beta}^2\widehat{\sigma}_{t-2}^2 \\
&\approx \widehat{\omega}(1+\widehat{\beta}) + (1-\widehat{\beta})(\varepsilon_{t-1}^2 + \widehat{\beta}\varepsilon_{t-2}^2) + \widehat{\beta}^2\widehat{\sigma}_{t-2}^2 \qquad (13.39) \\
&\approx \ ... \\
&\approx \widehat{\omega}(1+\widehat{\beta}+\widehat{\beta}^2+...) + (1-\widehat{\beta})(\varepsilon_{t-1}^2 + \widehat{\beta}\varepsilon_{t-2}^2 + \widehat{\beta}^2\varepsilon_{t-3}^2...) \\
&\approx \frac{\widehat{\omega}}{1-\widehat{\beta}} + \frac{\varepsilon_{t-1}^2 + \widehat{\beta}\varepsilon_{t-2}^2 + \widehat{\beta}^2\varepsilon_{t-3}^2 + ...}{1+\widehat{\beta}+\widehat{\beta}^2+...} \qquad (13.40) \\
&\approx \frac{\varepsilon_{t-1}^2 + \widehat{\beta}\varepsilon_{t-2}^2 + \widehat{\beta}^2\varepsilon_{t-3}^2 + ...}{1+\widehat{\beta}+\widehat{\beta}^2+...} \qquad (13.41)
\end{aligned}$$

where (13.39) substitutes $1 - \widehat{\beta}$ for $\widehat{\alpha}$ since $\widehat{\alpha} + \widehat{\beta} \approx 0.995$, (13.40) uses the fact that $1 + \widehat{\beta} + \widehat{\beta}^2 + ... = 1/(1-\widehat{\beta})$ while (13.41) neglects the constant $\widehat{\omega}/(1-\widehat{\beta})$, since $\widehat{\omega}$ is very small.

The approximation (13.41) is in fact an exponential smoother, the exponentially weighted moving average (EWMA) applied to $\varepsilon_t^2$. Forecasting with EWMA is optimal in the mean-square error sense for the state space model

$$x_t = \mu_t + \eta_t \tag{13.42}$$
$$\mu_t = \mu_{t-1} + \nu_t \tag{13.43}$$

where

- $\eta_t$ and $\nu_t$ are i.i.d. sequences
- $\mathrm{E}[\eta] = 0$ and $\mathrm{E}[\nu] = 0$.

Assuming

$$r_t = \sigma_t Z_t, \qquad \mathrm{E}[Z_t] = 0, \qquad \mathrm{E}\left[Z_t^2\right] = 1, \tag{13.44}$$

the GARCH (1,1) model can be written in the form of (13.42):

$$r_t^2 = \sigma_t^2 + \sigma_t^2(Z_t^2 - 1) = \sigma_t^2 + \eta_t \qquad \text{with} \qquad E[\eta_t] = 0. \tag{13.45}$$

where $\sigma_t$ is a deterministic, smooth function of time. Here, we assume the unconditional variance is time-varying and that the returns are independent but not i.i.d. Equation (13.43) incorporates the uncertainty about the form of the model.

Similarly, the non-parametric regression model shares a closely-related setup, sharing the state model representation and ability to incorporate uncertainty about the form of the model. In this case, the uncertainty is handled by modelling the signal $\mu$ as a deterministic function of time.

From non-parametric regression literature, the time-varying unconditional variance is estimated as:

$$\widehat{\sigma}_{t;h}^2 = \sum_{k=1}^{n} W_k(t;h) r_k^2 \tag{13.46}$$

where $h$ is the bandwidth, $K$ a kernel and

$$W_k(t;h) = K\left(\frac{t-k}{h}\right) \Big/ \sum_{k=1}^{n} K\left(\frac{t-k}{h}\right). \tag{13.47}$$

Together, these two equations are from Nadaraya-Watson (zero-degree local polynomial) kernel estimate of $\sigma_t^2$.

Empirical tests of the S&P 500 data from 16 May, 1995 to 29 April, 2003 find that the paths of the GARCH (1,1) estimate $\widehat{\sigma}_t^2$ and the non-parametric

estimate (13.46) closely follow each other. Furthermore, there were no statistically significant differences between GARCH (1,1) and a non-stationary, non-parametric regression approach to next-day volatility forecasting.

However, one possible critique of the non-parametric approach is that it lacks any dynamics. It is assumed that volatility is driven by unstructural, exogenous factors. Since not much is known about these factors, the current volatility level is forecasted to be the return volatility for the close future. Near future returns are modelled as i.i.d. with a variance equal to today's estimate.

In contrast, the GARCH (1,1) model contains an endogenous mechanism for the volatility process which is capable of foreseeing future developments in the movements of prices. Therefore, it can be argued that GARCH (1,1)'s vision of the future makes it a preferable choice.

### Modelling Performance

As previously mentioned, the GARCH (1,1) process is assumed to be a realistic data generating process for financial returns. According to this assumption, given a sample of data with parameters estimated on a reasonable long sub-sample, these parameters should provide a good model for the entire sample. However, when using the estimated parameters from the S&P 500 returns from 16 May, 1995 to 29 April, 2003 to describe the data of the entire 11 727 observation sample from March 1957 to October 2003, the GARCH (1,1) model's simulated samples differ vastly from the real data. In fact, the GARCH (1,1) model greatly overestimates the variance for the long series of S&P returns. In 25 000 simulated samples, GARCH (1,1) produced an interval of variances [0.00014, 0.01526] while the true variance of the returns is 0.00008, see Starica (2003).

The inaccurate results of these simulations require a closer look at the GARCH (1,1) model's estimated unconditional variance. Under the assumption of weak stationarity ($\alpha + \beta < 1$), the unconditional variance of the GARCH (1,1) model is

$$\sigma^2 = \omega/(1 - \alpha - \beta). \tag{13.48}$$

Using this formula for the sub-sample of S&P 500 returns from 16 May, 1995 to 29 April, 2003, the standard deviation of the GARCH (1,1) estimated unconditional variance ranges from 1.5 to 5 times larger than the sample's true standard deviation. Hence, the issue of the GARCH (1,1) model's inability to produce accurate unconditional point estimates has implications in its data forecasting abilities.

### Forecasting Performance

Assuming that GARCH (1,1) is a data generating process with $\alpha+\beta < 1$, the Mean Square Error (MSE) forecast for $E[r_{t+p}^2]$, the return variance $p$ steps ahead, is:

$$\sigma_{t+p}^{2,GARCH} = \mathrm{E}_t[r_{t+p}^2] = \sigma^2 + (\alpha+\beta)^p(\sigma_t^2 - \sigma^2), \tag{13.49}$$

where $\sigma^2$ is the unconditional variance defined in (13.48). Since $\alpha+\beta < 1$, for large values of $p$, the second term on the right-hand side of the equation goes to zero so that the forecast $\sigma_{t+p}^{2,GARCH}$ is equal to the unconditional variance $\sigma^2$. The minimum MSE forecast for the variance of the next $p$ aggregated returns is:

$$\overline{\sigma}_{t,p}^{2,GARCH} = \mathrm{E}[r_{t+1} + ... + r_{t+p}]^2 = \sigma_{t+1}^{2,GARCH} + ... + \sigma_{t+p}^{2,GARCH}. \tag{13.50}$$

As a benchmark (BM) for volatility forecasting, a simple non-stationary model (13.44) is used. As previously discussed, since this model does not specify any dynamics for the variance, future observations are modelled as i.i.d. with constant variance $\widehat{\sigma}_t^2$, which is an estimate of $\sigma_t^2$. Here, the sample variance of the previous year of returns will be used to estimate $\sigma_t^2$:

$$\widehat{\sigma}_t^2 = \frac{1}{250}\sum_{i=1}^{250} r_{t-i+1}^2. \tag{13.51}$$

Therefore, the forecast for $\mathrm{E}[r_{t+p}^2]$ is

$$\sigma_{t+p}^{2,BM} = \widehat{\sigma}_t^2. \tag{13.52}$$

The forecast for the variance of the next $p$ aggregated returns, $E[r_{t+1} + ... + r_{t+p}]^2$, is

$$\overline{\sigma}_{t,p}^{2,BM} = p\,\widehat{\sigma}_t^2 = \frac{p}{250}\sum_{i=1}^{250} r_{t-i+1}^2. \tag{13.53}$$

In order to evaluate forecasting performance, realised volatility $\overline{r}_{t,p}$ in the interval $[t+1, t+p]$ is defined by

$$\overline{r}_{t,p}^2 = \sum_{i=1}^{p} r_{t+i}^2. \tag{13.54}$$

The following formula is preferred for calculating an accurate measure of mean square error (MSE)

$$\sum_{t=1}^{n}(r_{t+p}^2 - \sigma_{t+p}^{2,*})^2 \tag{13.55}$$

where * represents BM or GARCH.

By initially estimating the GARCH (1,1) model on the first number of data points and then continuously re-estimating using a specified interval of time, simultaneously estimating $\sigma_t^2$ using (13.51), (13.50) and (13.53) can be used to forecast volatility for the year to come ($p = 1, \ldots, 250$). When forecasting the GARCH (1,1) volatility of S&P returns between 3 May, 1999 and 29 April, 2003, the effects of the over-estimation of conditional variance shown above are evident as GARCH (1,1) provides poor longer horizon forecasts for this period. In fact, the simple non-parametric, non-stationary model produced better forecasting results. Moreover, MSE analysis shows an increasing $\text{MSE}^{GARCH}/\text{MSE}^{BM}$ ratio over the forecast horizon. These results lead one to wonder if GARCH (1,1) always over-estimates unconditional variance or if the S&P sample used here is unique in some way. The following sections investigate this issue as the GARCH (1,1) model is applied to a longer sample of S&P data.

### Forecasting Performance of a Longer Sample of S&P Data

At this point, it is important to examine the options that one has when using the stationary GARCH (1,1) model. These choices depend on the working assumptions made about the data to be modelled. If the data is assumed to be stationary, a GARCH (1,1) process would be used as a true data generating process for the entire data set. In this case, it is best to use as much data as possible in order to draw any statistical conclusions or to make any forecasts.

However, if the data if assumed to be non-stationary, then the GARCH (1,1) process might be used as a local approximation for a true data generating process. Given this condition, the parameters of the model should be periodically re-estimated on a moving window. Only data from the recent past should be used to calibrate the model to be used for forecasting. Making any statistical statements based on long samples should be avoided. In the situation of non-stationary data, it is unclear whether GARCH (1,1) can produce accurate long-horizon forecasts.

**Hypothesis: Returns are Stationary**

Under the working assumption of stationary data, GARCH (1,1) estimated parameters should be unchanging if they are estimated on an increasing sample size or on a window that moves through the data. When testing a hypothesis of stationarity, if significant changes in the estimated parameters outside of the confidence interval are detected, then the hypothesis should be rejected and the model should not be considered to be a true data generating process.

For this hypothesis testing, Starica uses a much longer sample of S&P data from 4 March, 1957 to 9 October, 2003, re-estimating the data every 50 observations, both on a moving window of 2000 past observations and on an increasing sample containing all past observations. However, the hypothesis of stationarity was strongly rejected as the estimated parameters changed significantly over time and did a poor job of remaining in their confidence intervals. A similar analysis was performed on returns from the Dow Jones Industrial index for the same period. The hypothesis was also rejected for the Dow Jones series. From this evidence, it can be concluded that the GARCH (1,1) process is not a data generating process for the returns of the S&P 500 index between 4 March, 1957 to 9 October, 2003.

The significant changes in the value of the estimated parameters suggests that the data is not stationary, particularly that the unconditional variance might be time-varying. This issue will be examined as a hypothesis of non-stationarity next.

**Hypothesis: Returns are Non-stationary**

The assumption that long range S&P data may be non-stationary due to a time-varying unconditional variance might explain the poor point estimates described above. Under closer examination of the sum $\alpha+\beta$, estimated from a moving window of 2000 S&P returns from 1957 to 2003, IGARCH effects are present in the period from 1997 to 2003 as $\alpha + \beta < 1$ for this period. An IGARCH effect is likely to cause an explosion of estimated unconditional variance when fitting GARCH (1,1) processes to samples that end in this time period, see equation (13.50).

In fact, a comparison of the GARCH(1,1) and sample standard deviations reveals a good agreement between the two estimates for the entire period except from 1997 to 2003 when the IGARCH effect becomes strongly statistically significant. Analysis of the Dow Jones Industrial Index shows similar findings, see Starica (2003).

The strong IGARCH effect in the late 1990's can be attributed to a sharp

change in unconditional variance. In the paper by Mikosch and Starica (2004), it is proved, theoretically and empirically, that sharp changes in unconditional variance can cause the IGARCH effect. Similarly, non-parametric modelling of the unconditional standard deviation of both the S&P 500 and Dow Jones also reveal that the level of volatility triples from 5% in 1993-1994 to 15% in 2000-2003.

#### Implications on Long Horizon Volatility Forecasting

As discussed above, the coefficients of a GARCH (1,1) model can change significantly over time. Moreover, these parameters vary greatly when looking at the S&P 500 and Dow Jones Industrial Index, both of which are major financial indices.

In order to take a closer look at this issue, one can look at a comparison of the forecasting performances of GARCH (1,1) versus a simple approach based on historical volatility, using a similar approach as described above, see Starica (2003). From analysis of $\text{MSE}^{GARCH}$ and $\text{MSE}^{BM}$, it is evident that for shorter horizons (60 days ahead for the locally estimated model and 40 days ahead for the model using all past observations) $\text{MSE}^{GARCH} < \text{MSE}^{BM}$, however beyond this short forward-looking horizon, $\text{MSE}^{GARCH} > \text{MSE}^{BM}$, meaning that for longer horizons, the approach based on historical volatility performed better. The GARCH (1,1) approach becomes significantly more volatile during the late 1990's, the period in which the IGARCH effect is present, than does the historical volatility method. GARCH (1,1)'s poor performance is also seen for the Dow Jones series of returns, leading one to question its applicability in long-term volatility forecasting.

### 13.3.2 Next-Day Volatility Forecasting for DAX Returns

Based on Starica's example, this section compares GARCH (1,1) next-day volatility forecasting with a non-parametric estimation very similar to the method described in Section 13.3.1 for DAX returns from 1 January, 1992 to 31 December, 2004. Figure 13.9 shows a plot of the daily logarithmic DAX returns (first differences of the logarithms of the daily closing prices).

Next, a GARCH (1,1) process (13.38) was fitted to the DAX data to provide a next-day volatility forecast as seen in Figure 13.10. The estimated GARCH (1,1) parameters are

$$\widehat{\omega} = 1.4436 \times 10^{-6}, \qquad \widehat{\alpha} = 0.076342, \qquad \widehat{\beta} = 0.91711. \tag{13.56}$$

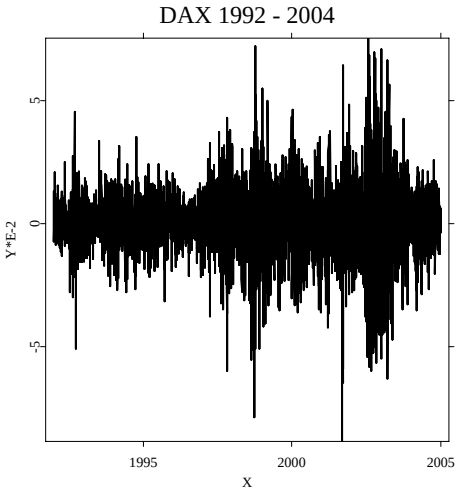


Figure 13.9: DAX returns from 1992 to 2004.

SFEdaxret

This indicates the presence of an IGARCH effect which can most likely be attributed to this DAX data being non-stationary with a changing and overestimated unconditional variance. However, to confirm this assumption, further analysis of the unconditional variance is necessary.

Figure 13.11 shows the estimated volatility resulting from using the non-parametric model based on historical volatility. This non-parametric estimate was calculated using a Nadaraya-Watson kernel estimate. The optimal bandwidth was calculated by choosing the value of $h$ that minimises the final prediction error, see Härdle, Müller, Sperlich and Werwatz (2004). It is evident when comparing the non-parametric estimate in Figure 13.11 to the GARCH (1,1) estimate in Figure 13.10, that the paths of these volatility processes are quite similar.

From this analysis of DAX returns, it is observed that the late 1990's as well as 2002-2004 were all periods of high volatility. This can be seen in the plot of the DAX returns, and it is captured by both the GARCH (1,1) and non-parametric next-day volatility estimates. Moreover, these results are very similar to Starica's study of the S&P 500 returns, demonstrating the correlation between the returns of the securities represented by both indices.

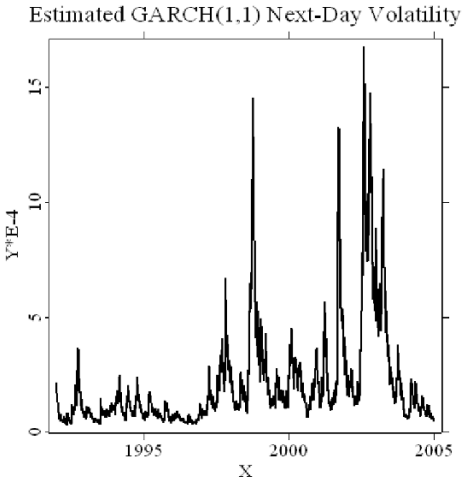


Figure 13.10: Estimated GARCH next-day volatility for DAX returns from 1992-2004.

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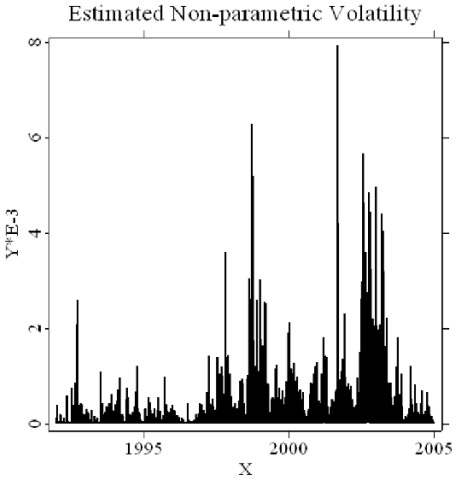


Figure 13.11: Estimated volatility of DAX returns from 1992-2004 using a non-parametric model based on historical volatility.

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## 13.4 Multivariate GARCH Models

The generalisation of univariate GARCH models to the multivariate case is straightforward. For the error term $\varepsilon_t$ of a $d$-dimensional time series model we assume that the conditional mean is zero and the conditional covariance matrix is given by the positive definite $(d \times d)$ matrix $H_t$, i.e.,

$$\varepsilon_t = H_t^{1/2} \xi_t \tag{13.57}$$

with i.i.d. innovation vector $\xi_t$, whose mean is zero and covariance matrix equals the identity matrix $I_d$. As in the univariate case, $H_t$ depends on lagged error terms $\varepsilon_{t-i}$, $i = 1, \ldots, q$, and on lagged conditional covariance matrices $H_{t-i}$, $i = 1, \ldots, p$,. As we will see shortly, the general case with arbitrary dependencies can lead to very complex structures that may be too difficult to deal with in practice. The reduction of the dimension of the parameter space is therefore often tried. Below, we first discuss a general specification and then a popular restriction, the BEKK model. We will also briefly sketch a model that assumes constant conditional correlations.

### 13.4.1 The Vec Specification

Let vech$(\cdot)$ denote the operator that stacks the lower triangular part of a symmetric $d \times d$ matrix into a $d^* = d(d+1)/2$ dimensional vector. Furthermore we will use the notation $h_t = \text{vech}(H_t)$ and $\eta_t = \text{vech}(\varepsilon_t \varepsilon_t^\top)$. The *Vec specification* of a multivariate GARCH$(p, q)$ model is then given by

$$h_t = \omega + \sum_{i=1}^{q} A_i \eta_{t-i} + \sum_{j=1}^{p} B_j h_{t-j}, \tag{13.58}$$

where $A_i$ and $B_j$ are parameter matrices with each one containing $(d^*)^2$ parameters. The vector $\omega$ represents constant components of the covariances and contains $d^*$ parameters.

For the bivariate case and $p = q = 1$ we can write the model explicitly as

$$\begin{pmatrix} h_{11,t} \\ h_{12,t} \\ h_{22,t} \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t-1}^2 \\ \varepsilon_{1,t-1}\varepsilon_{2,t-1} \\ \varepsilon_{2,t-1}^2 \end{pmatrix}$$
$$+ \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} h_{11,t-1} \\ h_{12,t-1} \\ h_{22,t-1} \end{pmatrix}$$

By rearranging terms, we can write the second order process $\eta_t$ as a vector autoregressive moving average (VARMA) process of order $(\max(p,q),p)$,

$$\eta_t = \omega + \sum_{i=1}^{\max(p,q)} (A_i + B_i)\eta_{t-i} - \sum_{j=1}^{p} B_j u_{t-j} + u_t, \tag{13.59}$$

where $u_t = \eta_t - h_t$ is a vector white noise process, i.e., $\mathsf{E}[u_t] = 0$, $\mathsf{E}[u_t u_t^\top] = \Sigma_u$ und $\mathsf{E}[u_t u_s^\top] = 0$, $s \neq t$. In (13.59) we set $A_{q+1} = \ldots = A_p = 0$ if $p > q$ and $B_{p+1} = \ldots = B_q = 0$ if $q > p$. Often the VARMA representation of multivariate GARCH models simplifies the derivation of stochastic properties, as one can refer to known results of the VARMA literature.

In the Vec representation (13.58), the multivariate GARCH$(p,q)$ process $\varepsilon_t$ is covariance stationary if, and only if, all eigenvalues of the matrix

$$\sum_{i=1}^{\max(p,q)} (A_i + B_i)$$

are smaller than one in modulus, see Engle and Kroner (1995). In that case, the unconditional covariance matrix is given by

$$\sigma = \operatorname{vech}(\Sigma) = \left( I_{d^*} - \sum_{i=1}^{\max(p,q)} (A_i + B_i) \right)^{-1} \omega. \tag{13.60}$$

In order to illustrate the prediction of volatility, let us consider the following often used GARCH(1,1) model. The optimal prediction with respect to the mean squared prediction error is the conditional expectation of volatility. Due to the law of iterated expectations, the $k$-step prediction of $\eta_{t+k}$ is identical to the $k$-step prediction of $h_{t+k}$, that is,

$$\mathsf{E}[\eta_{t+k} \mid \mathcal{F}_t] = \mathsf{E}[\mathsf{E}(\eta_{t+k} \mid \mathcal{F}_{t+k-1}) \mid \mathcal{F}_t] = \mathsf{E}[h_{t+k} \mid \mathcal{F}_t].$$

Having information up to time $t$, the predictions for the next three time periods are given by

$$\begin{aligned}
\mathsf{E}[\eta_{t+1} \mid \mathcal{F}_t] &= h_{t+1} \\
\mathsf{E}[\eta_{t+2} \mid \mathcal{F}_t] &= \omega + (A+B)h_{t+1} \\
\mathsf{E}[\eta_{t+3} \mid \mathcal{F}_t] &= (I_{d^*} + A + B)\omega + (A+B)^2 h_{t+1},
\end{aligned}$$

and it can be seen that in general, the $k$-step prediction with $k \geq 2$ is given by

$$\mathsf{E}[\eta_{t+k} \mid \mathcal{F}_t] = \left\{ I_{d^*} + (A+B) + \ldots + (A+B)^{k-2} \right\} \omega + (A+B)^{k-1} h_{t+1}.$$

This converges to the unconditional covariance matrix $\sigma = (I_{d^*} - A - B)^{-1}\omega$ if, and only if, the process is covariance stationary.

In the bivariate case ($d = 2$) and with $p = q = 1$, there are already 21 parameters that characterise the dynamics of volatility. In order to obtain a feasible model for empirical work, one often imposes restrictions on the parameter matrices of the Vec model. Bollerslev, Engle and Wooldridge (1988) propose using diagonal parameter matrices such that the conditional variance of one variable only depends on lagged squared values of the same variable, and the conditional covariances between two variables only depend on lagged values of the cross-products of these variables. This model substantially reduces the number of parameters (in the above case from 21 to 9), but potentially important causalities are excluded.

For parameter estimation the Quasi Maximum Likelihood Method (QML) is suitable. The conditional likelihood function for a sample time series of $n$ observations is given by $\log L = \sum_{t=1}^{n} l_t$ with

$$l_t = -\frac{d}{2}\log(2\pi) - \frac{1}{2}\log\{\det(H_t)\} - \frac{1}{2}\varepsilon_t^\top H_t^{-1}\varepsilon_t. \tag{13.61}$$

If the conditional distribution of $\varepsilon_t$ is not normal, then (13.61) is interpreted as a quasi likelihood function, which serves merely as a target function in the numerical optimisation, but which does not say anything about the true distribution. In the multivariate case, the QML estimator is consistent and asymptotically normal under the main assumptions that the considered process is strictly stationary and ergodic with a finite eighth moment. Writing all parameters in one vector, $\theta$, we obtain the following standard result.

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{\mathcal{L}} \mathrm{N}(0, J^{-1}IJ^{-1}), \tag{13.62}$$

where $I$ is the expectation of outer product of the score vector (i.e., the vector $\partial l_t/\partial\theta$), and $J$ the negative expectation of the Hessian (i.e., the matrix of second derivatives). In the case of a normal distribution, we have $I = J$ and the asymptotic distribution simplifies to

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{\mathcal{L}} \mathrm{N}(0, J^{-1}). \tag{13.63}$$

In other words, these results are completely analogous to the univariate case, but the analytical expressions for $I$ and $J$ become much more complicated. Of course one can also determine $I$ and $J$ numerically, but this can lead to unreliable results, especially for $J$, in the multivariate case.

In empirical work one often finds that estimated standardised residuals are not normally distributed. In this case the QML likelihood function would be

mis-specified and provides only consistent, not efficient parameter estimators. Alternatively, one can assume that the true innovation distribution is given by some specific non-normal parametric distribution, but in general this does not guarantee that parameter estimates are consistent in the case that the assumption is wrong.

### 13.4.2 The BEKK Specification

Engle and Kroner (1995) discuss the following specification of a multivariate GARCH model.

$$H_t = C_0 C_0^\top + \sum_{k=1}^{K} \sum_{i=1}^{q} A_{ki}^\top \varepsilon_{t-i} \varepsilon_{t-i}^\top A_{ki} + \sum_{k=1}^{K} \sum_{j=1}^{p} B_{kj}^\top H_{t-j} B_{kj}. \qquad (13.64)$$

In (13.64), $C_0$ is a lower triangular matrix and $A_{ki}$ and $B_{ki}$ are $d \times d$ parameter matrices. For example, in the bivariate case with $K = 1$, $p = 1$ and $q = 0$, the conditional variance of $\varepsilon_{1t}$ can be written as

$$h_{11,t} = c_{11}^2 + a_{11}^2 \varepsilon_{1t}^2 + a_{12}^2 \varepsilon_{2t}^2 + 2 a_{11} a_{12} \varepsilon_{1t} \varepsilon_{2t}$$

and the conditional covariance as

$$h_{12,t} = c_{11} c_{21} + a_{11} a_{21} \varepsilon_{1t}^2 + a_{12} a_{22} \varepsilon_{2t}^2 + (a_{12} a_{21} + a_{11} a_{22}) \varepsilon_{1t} \varepsilon_{2t}$$

The so-called BEKK specification in (13.64) guarantees under weak assumptions that $H_t$ is positive definite. A sufficient condition for positivity is for example that at least one of the matrices $C_0$ or $B_{ki}$ have full rank and the matrices $H_0, \ldots, H_{1-p}$ are positive definite. The BEKK model allows for dependence of conditional variances of one variable on the lagged values of another variable, so that causalities in variances can be modelled. For the case of diagonal parameter matrices $A_{ki}$ and $B_{ki}$, the BEKK model is a restricted version of the Vec model with diagonal matrices.

Due to the quadratic form of the BEKK model, the parameters are not identifiable without further restriction. However, simple sign restrictions will give identifiability. For example, in the often used model $K = 1$ and $p = q = 1$, it suffices to assume that the upper left elements of $A_{11}$ and $B_{11}$ are positive. The number of parameters reduces typically strongly when compared to the Vec model. For the above mentioned case, the number of parameters reduces from 21 to 11.

For each BEKK model there is an equivalent Vec representation, but not vice versa, so that the BEKK model is a special case of the Vec model. To see this,

just apply the vech operator to both sides of (13.64) and define $\omega = L_d(C_0 \otimes C_0)^\top D_d \text{vech}(I_d)$, $A_i = \sum_{k=1}^K L_d(A_{ki} \otimes A_{ki})^\top D_d$, and $B_j = \sum_{k=1}^K L_d(B_{kj} \otimes B_{kj})^\top D_d$. Here $\otimes$ denotes the Kronecker matrix product, and $L_d$ and $D_d$ are the elementary elimination and duplication matrices. Therefore, one can derive the stochastic properties of the BEKK model by those of the Vec model. For the empirical work, the BEKK model will be preferable, because it is much easier to estimate while being sufficiently general.

### 13.4.3 The CCC Model

Bollerslev (1990) suggested a multivariate GARCH model in which all conditional correlation are constant and the conditional variances are modelled by univariate GARCH models. This so-called CCC model (constant conditional correlation) is not a special case of the Vec model, but belongs to another, nonlinear model class. For example, the CCC(1,1) model is given by

$$h_{ii,t} = \omega_i + \alpha_i \varepsilon_{i,t-1}^2 + \beta_i h_{ii,t-1},$$

$$h_{ij,t} = \rho_{ij} \sqrt{h_{ii,t} h_{jj,t}}$$

for $i, j = 1 \ldots, d$, and $\rho_{ij}$ equal to the constant correlation between $\varepsilon_{it}$ and $\varepsilon_{jt}$, which can be estimated separately from the conditional variances. The advantage of the CCC model is in the unrestricted applicability for large systems of time series. On the other hand, the assumption of constant correlation is possibly quite restrictive. For example, in the empirical analysis of financial markets one typically observes increasing correlation in times of crisis or in crash situations.

### 13.4.4 The DCC Model

Engle (2002) has introduced a generalisation of the CCC model that allows for time-varying correlations in a straightforward way. Similar to the CCC model the idea is to separate the modelling of volatilities and correlations and render estimation feasible in high dimensions. Again, volatilities $h_{ii,t}$ could be modelled in a first step by univariate GARCH(1,1) models as discussed in the previous section for the CCC model. Rather than assuming that the conditional correlation $\rho_{ij}$ between the $i$-th and $j$-th component is constant, it is now the $ij$-th element of the matrix

$$R_t = \text{diag}(Q_t)^{-1/2} Q_t \, \text{diag}(Q_t)^{-1/2}$$

with

$$Q_t = \Omega + \gamma v_{t-1} v_{t-1}' + \delta Q_{t-1},$$

| | Min. | Max. | Mean | Median | Std.Error |
|---|---|---|---|---|---|
| DEM/USD | −0.040 | 0.032 | $-4.718e-06$ | 0 | 0.0071 |
| GBP/USD | −0.047 | 0.039 | 0.000110 | 0 | 0.0070 |

Table 13.4: SFEmvol01

where $v_t = (\varepsilon_{1,t}/\sqrt{h_{11,t}}, \ldots, \varepsilon_{d,t}/\sqrt{h_{dd,t}})'$, $\Omega$ is a symmetric positive definite parameter matrix and $\gamma$ and $\delta$ are positive scalars. Note that by construction, $Q_t$ is positive definite and hence $R_t$ is a correlation matrix, i.e. it is positive definite and all elements on the diagonal are equal to one. A similar specification has been suggested by Tse and Tsui (2002).

### 13.4.5 An Empirical Illustration

We consider a bivariate exchange rates example, two European currencies, DEM and GBP, with respect to the US Dollar. The sample period is 01/01/1980 to 04/01/1994 with altogether $n = 3720$ observations. Figure 13.12 shows the time series of returns on both exchange rates. Table 13.4 provides some simple descriptive statistics of returns $\varepsilon_t$. Apparently, the empirical mean of both processes is close to zero.

As can be seen in Figure 13.12, the exchange rate returns follow a pattern that resembles a GARCH process: there is a clustering of volatilities in both series, and the clusters tend to occur simultaneously. This motivates an application of a bivariate GARCH model.

A first simple method to estimated the parameters of a BEKK model is the BHHH algorithm. This algorithm uses the first derivatives of the QML likelihood with respect to the 11 parameters that are contained in $C_0$, $A_{11}$ and $G_{11}$, recalling equation (13.64). As this is an iterative procedure, the BHHH algorithm needs suitable initial parameters. For the diagonal elements of the matrices $A_{11}$ and $B_{11}$, values between 0.3 and 0.9 are sensible, because this is the range often obtained in estimations. For the off-diagonal elements there is no rule of thumb, so one can try different starting values or just set them to zero. The starting values for $C_0$ can be obtained by the starting values for $A_{11}$ and $B_{11}$ using the formula for the unconditional covariance matrix and matching the sample covariance matrix with the theoretical version.

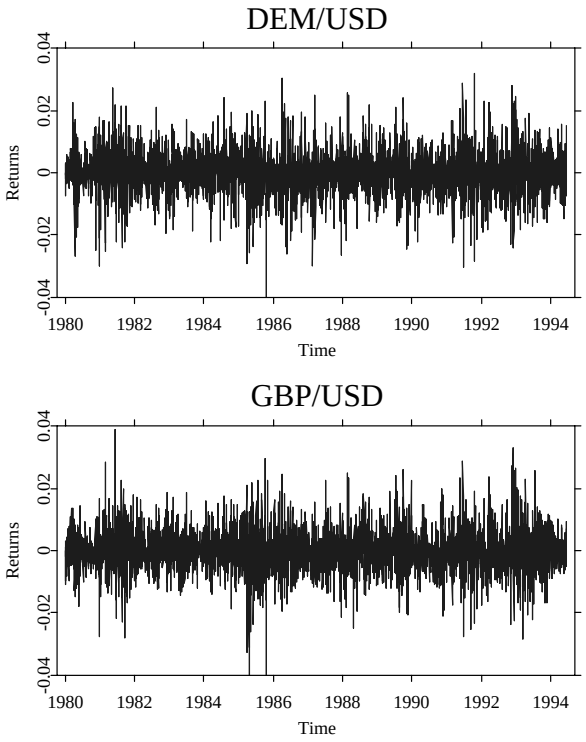


Figure 13.12: Exchange rate returns. SFEmvol01

For the bivariate exchange rate example, we obtain the following estimates:

$$\hat{\theta} = \begin{pmatrix} 0.00115 \\ 0.00031 \\ 0.00076 \\ 0.28185 \\ -0.05719 \\ -0.05045 \\ 0.29344 \\ 0.93878 \\ 0.02512 \\ 0.02750 \\ 0.93910 \end{pmatrix}$$

$$\sum_{t=1}^{n} l_t = -28599$$

SFEmvol02

the previous value represents the computed minimum of the negative log likelihood function. The displayed vector contains in the first three components the parameters in $C_0$, the next four components are the parameters in $A_{11}$, and the last four components are the parameters in $B_{11}$.

In this example we thus obtain as estimated parameters of the BEKK model:

$$C_0 = 10^{-3} \begin{pmatrix} 1.15 & 0.31 \\ 0.00 & 0.76 \end{pmatrix},$$
$$A_{11} = \begin{pmatrix} 0.282 & -0.050 \\ -0.057 & 0.293 \end{pmatrix}, \; B_{11} = \begin{pmatrix} 0.939 & 0.028 \\ 0.025 & 0.939 \end{pmatrix}. \tag{13.65}$$

Estimates for the conditional covariances are obtained by applying successively the difference equation (13.64), where the empirical covariance matrix

$$\hat{H}_0 = \frac{1}{n} \sum_{t=1}^{n} \varepsilon_t \varepsilon_t^\top$$

of the observations $\varepsilon_t$ is taken as initial value.

In Figure 13.13 estimated conditional variance and covariance processes are compared. The upper and lower plots show the variance of the DEM/USD and GBP/USD returns and the plot in the middle shows the estimated conditional covariance process. Apart from a very short period at the beginning of the sample, the covariance is positive and of significnat magnitude. This confirms our intuition of mutual dependence in exchange markets which motivated the use of the bivariate GARCH model.

SFEmvol03

The estimated parameters can also be used to simulate volatility. This can be done by drawing one realisation of a multivariate normal distribution with mean zero and variance $\hat{H}_t$ at every time step. With these realisations one updates $\hat{H}_t$ according to equation (13.64). Next, a new realisation is obtained by drawing from $\mathrm{N}(0, \hat{H}_{t+1})$, and so on. We will now apply this method with $n = 3000$. The results of the simulation in Figure 13.14 show similar patterns as in the original process (Figure 13.13). For a further comparison, we include two independent univariate GARCH processes fitted to the two exchange rate return series. This corresponds to a bivariate Vec representation with

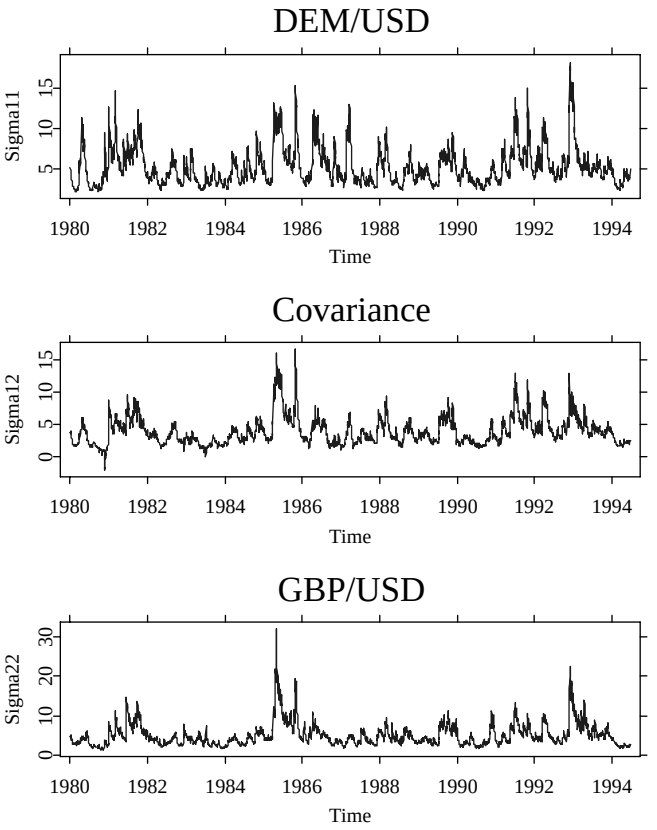


Figure 13.13: Estimated variance and covariance processes, $10^5 \hat{H}_t$.
SFEmvol02

diagonal parameter matrices. Obviously, both methods capture the clustering of volatilities. However, the more general bivariate model also captures a spill over effect, that is, the increased uncertainty in one of the returns due to increased volatility in the other returns. This has an important impact on the amplitude of volatility.

## 13.5 Recommended Literature

The empirical properties of financial market data, in particular the leptokurtosis and clustering of volatilities, have been investigated systematically by Mandelbrot (1963) and Fama (1965). ARCH Models were introduced by Engle (1982) and generalised to GARCH by Bollerslev (1986). For an excellent overview of ARCH models we refer to Gouriéroux (1997). An extensive discussion of the estimation of the tail exponent and the Hill estimator can be found in Embrechts, Klüppelberg and Mikosch (1997). A comparison of the efficiency of the Yule Walker estimator with ML estimators of ARCH and GARCH models is given in Maercker (1997).

The theory of QML estimation was developed by White (1982) and Gouriéroux, Monfort and Trognon (1984). Weiss (1986) applied the theory to ARCH models, Lee and Hansen (1994) and Lumsdaine (1996) to GARCH models. Bollerslev and Wooldridge (1992) considered QML estimation for general conditionally heteroscedastic models. The idea of the *smooth transition* specification stems from Teräsvirta (1994) who used it for AR models. It was applied to GARCH models by Lubrano (1998). Hafner and Herwartz (2000) discussed various methods to test for the significance of the AR parameter under conditional heteroscedasticity. Moreover, they compare empirical results of AR models versus ARCH-M models, applied to several German stock returns.

Starica (2003) examines some shortfalls of GARCH, primarily its inability to function as a true data generating process and accurate forecasting tool as well as GARCH's overestimation of the unconditional variance and the existence of the IGARCH effect.

GARCH(1,1) model can be written in the form of the state space model. A closely related set-up, which shares with the state model representation the ability to incorporate our uncertainty about the form of the model is that of the non-parametric regression, which is discussed in Wand and Jones (1995).

In the multivariate case, Jeantheau (1998) has shown the consistency of the QML estimator, and Comte and Lieberman (2003) derived asymptotic normality. Analytical expressions for the score and Hessian matrices are provided by Hafner and Herwartz (2003). A recent survey of multivariate GARCH models is provided by Bauwens, Laurent and Rombouts (2005). It discusses the various specifications, estimation, inference, and applications.

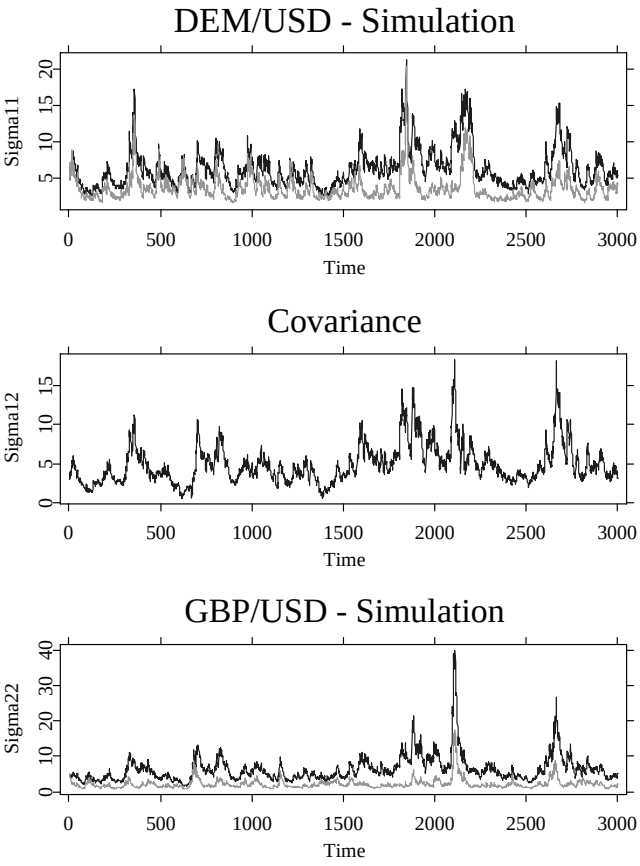


Figure 13.14: Simulated variance and covariance processes with a bivariate (blue) and two univariate (green) GARCH processes, $10^5 \hat{H}_t$. SFEmvol03

# 14 Non-parametric Concepts for Financial Time Series

With the analysis of (financial) time series, one of the most important goals is to produce forecasts. Using past observed data one can make some statements about the future mean, the future volatility, etc., for example, one would like to estimate the expectation and variance of the underlying process conditional on the past. One method of producing such estimates will be introduced in this chapter.

Let $(Y_t)$, $t = 0, 1, 2, \ldots$, be a time series. We consider a nonlinear autoregressive heteroscedastic model of the form

$$Y_t = f(Y_{t-1}) + s(Y_{t-1})\xi_t, \quad t = 1, 2, \ldots \tag{14.1}$$

Here the innovations $\xi_t$ are i.i.d. random variables with $\mathsf{E}[\xi_t] = 0$ and $\mathsf{E}[\xi_t^2] = 1$, $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $s : \mathbb{R} \longrightarrow (0, \infty)$ are unknown functions, and $Y_0$ is independent of $(\xi_t)$. Under these assumptions and according to Theorem 3.1 it holds that:

$$\mathsf{E}[Y_t \,|\, Y_{t-1} = x] = f(x) + \mathsf{E}[s(Y_{t-1})\xi_t \,|\, Y_{t-1} = x] = f(x) + s(x)\mathsf{E}[\xi_t] = f(x),$$

where in the second to last equation the independence of $\xi_t$ and $Y_{t-1}$ is used. A similar calculation gives $s^2(x) = \mathrm{Var}[Y_t \,|\, Y_{t-1} = x]$. The unknown functions $f$ and $s$ describe the conditional mean and the conditional volatility of the process, which we want to estimate.

With the specific choice $f(x) = \alpha x$ and $s = \sigma > 0$ the process $Y_t$ is an AR(1) process. Every strong ARCH(1) process $(Y_i)$ satisfies the model (14.1). In this case $f = 0$, and it holds that $s(x) = \sqrt{\omega + \alpha x^2}$ with the parameters $\omega > 0$ and $\alpha \geq 0$, compare Definition 13.1. With respect to the structure of the conditional mean and the conditional variance, the model above is another broad generalisation of the (strong) ARCH models. The advantage of this *nonparametric* Ansatz is that the model contains no structural assumptions about the functions $f$ and $s$, since such assumptions are often not supported by observations in the data.

Closely related to our model is the Qualitative Threshold ARCH model (QTARCH) studied in Gouriéroux and Monfort (1992), which for the case of

one lag (QTARCH(1)) is a special case of (14.1), where the unknown functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $s : \mathbb{R} \longrightarrow (0, \infty)$ take the form of step functions - see (13.32). On the other hand (14.1) can also be described under certain regularity assumptions on $f$ and $s$ as a limit model of the QTARCH(1) models when $J \to \infty$, in that $f$ and $s$ are approximated with elementary functions.

The work of Gouriéroux and Monfort is the first to consider the conditional mean and the conditional variance *together* in a nonparametric model. The applications of this idea introduced here are taken from Härdle and Tsybakov (1997), and are also considered independently in Franke, Kreiss and Mammen (2002). In the following we will construct a class of estimators based on the local polynomial regression for the conditional volatility $v(x) = s^2(x)$ and the conditional mean $f(x)$ of the time series $(Y_i)$ under the model assumptions (14.1).

In addition to the model assumptions (14.1) certain regularity assumptions, although no structural assumptions, on $f$ and $s$ will be made. As the main result of this chapter we will show, this combined estimation of the conditional expectation and the conditional volatility is asymptotically normally distributed.

## 14.1 Nonparametric Regression

In this section we introduce several basic terms and ideas from the theory of nonparametric regressions and explain in particular the method of local polynomial regression. To conclude we explain how this can be applied to (financial) time series. A detailed representation can be found in Härdle et al. (2004).

In nonparametric regression one is interested in the (functional) relationship between an explanatory variable $X$ and a dependent variable $Y$, i.e., one is interested in obtaining an estimation for the unknown function $m(x) = \mathsf{E}[Y \mid X = x]$. In doing this, in contrast to parametric statistics, no special assumptions on the form of the function $m$ is made. Only certain regularity and smoothing assumptions are made about $m$.

One way to estimate $m$ is to use the method of local polynomial regression (LP Method). The idea is based on the fact that the function $m$ can be locally approximated with a Taylor polynomial, i.e., in a neighborhood around a given point $x_0$ it holds that

$$m(x) \approx \sum_{k=0}^{p} \frac{m^{(k)}(x_0)}{k!}(x - x_0)^k. \tag{14.2}$$

In order to find an estimate for $m$ at point $x_0$, one therefore tries to find a polynomial based on observations $(X_1, Y_1), \ldots, (X_n, Y_n)$ that is a good approximation of $m$ around $x_0$. As a measure of the quality of the approximation one usually chooses a LS criterion, i.e., one wants to minimise the expression

$$\sum_{i=1}^{n} \Big\{ Y_i - \sum_{j=0}^{p} \beta_j (X_i - x_0)^j \Big\}^2 \tag{14.3}$$

with respect to $\beta = (\beta_0, \ldots, \beta_p)^\top$. Since the representation (14.2) holds only locally, one still has to take into consideration that some of the observations $X_i$ may not lie close enough to $x_0$ and thus (14.2) no longer applies to them. One must then sufficiently localise the observations, i.e., only consider those observations that lie close enough to $x_0$.

One of the classical methods for localisation is based on weighting the data with the help of a kernel. A kernel is a function $K : \mathbb{R} \longrightarrow [0, \infty)$ with $\int K(u)\, du = 1$. The most useful kernels are also symmetric and disappear outside of a suitable interval around the zero point.

If $K$ is a kernel and $h > 0$, then the kernel $K_h$

$$K_h(u) = \frac{1}{h} K\big(\frac{u}{h}\big)$$

is re-scaled with the *bandwidth* $h$, which again integrates to 1. If, for example, the initial kernel $K$ disappears outside of the interval $[-1, 1]$, then $K_h$ is zero outside of the interval $[-h, h]$. By weighting the $i-$th term in (14.3) with $K_h(x - X_i)$, one has a minimisation problem which, due to the applied localisation, can be formulated to be independent of the point $x_0$. The coefficient vector $\hat{\beta} = \hat{\beta}(x) = (\hat{\beta}_0(x), \ldots, \hat{\beta}_p(x))^\top$ that determines the polynomial of the point $x$ is thus given by

$$\hat{\beta} = \arg\min_{\beta} \sum_{i=1}^{n} \Big\{ Y_i - \sum_{j=0}^{p} \beta_j (x - X_i)^j \Big\}^2 K_h(x - X_i). \tag{14.4}$$

It is obvious that $\hat{\beta}$ depends heavily on the choice of kernel and the bandwidth. Different methods for determining $K$ and $h$ are introduced in Härdle et al. (2004).

With the representation

$$\mathbf{X} = \begin{pmatrix} 1 & X_1 - x & (X_1 - x)^2 & \dots & (X_1 - x)^p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_n - x & (X_n - x)^2 & \dots & (X_n - x)^p \end{pmatrix}, \mathbf{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix},$$

$$\mathbf{W} = \begin{pmatrix} K_h(x - X_1) & & 0 \\ & \ddots & \\ 0 & & K_h(x - X_n) \end{pmatrix}$$

the solution $\hat{\beta}$ to the weighted least squares problem (14.4) can be explicitly written as

$$\hat{\beta}(x) = \left(\mathbf{X}^\top \mathbf{W} \mathbf{X}\right)^{-1} \mathbf{X}^\top \mathbf{W} \mathbf{Y} \tag{14.5}$$

The estimation $\hat{m}(x)$ for $m(x)$ can be obtained only by calculating the approximating polynomial at $x$:

$$\hat{m}(x) = \hat{\beta}_0(x). \tag{14.6}$$

The remaining components of $\hat{\beta}(x)$, due to equations (14.2) and (14.3) deliver estimators for the derivatives of $m$: $\hat{m}^{(j)}(x) = j!\,\hat{\beta}_j(x), j = 1, ..., p$, which will not be discussed in further detail here. In the special case where $p = 0$, $\hat{m}(x)$ is a typical kernel estimator of Nadaraya-Watson type, see Härdle (1990).

The similarly derived method of local polynomial approximation, or LP method for short, will now be applied to a time series $(Y_i)$. As mentioned before, one is most interested in creating forecasts.

For the simplest case a one-step ahead forecast means that the functional relationship between $Y_{i-1}$ and a function $\lambda(Y_i)$ of $Y_i$ will be analysed, i.e., we want to obtain an estimate for the unknown function

$$m(x) = \mathsf{E}\big[\lambda(Y_i) \,|\, Y_{i-1} = x\big].$$

In order to apply the LP Method mentioned above, consider a given sample $Y_0, \dots, Y_n$ as observations of the form $(Y_0, Y_1), \dots, (Y_{n-1}, Y_n)$. The process $(Y_i)$ must fulfil certain conditions, so that these observations are identically distributed and in particular so that the function $m$ is independent of the time index $i$. Such is the case when $(Y_i)$ is stationary. By substituting $X_i = Y_{i-1}$ into (14.4) and replacing $Y_i$ with $\lambda(Y_i)$, we obtain in this situation

$$\hat{\beta} = \arg\min_{\beta} \sum_{i=1}^{n} \Big\{ \lambda(Y_i) - \sum_{j=0}^{p} \beta_j (x - Y_{i-1})^j \Big\}^2 K_h(x - Y_{i-1}), \tag{14.7}$$

and the estimate for $m(x)$ is again given by $\hat{\beta}_0(x)$.

## 14.2 Construction of the Estimator

The LP method introduced in the previous section will now be applied under the assumption of a nonparametric autoregressive model of the form (14.1) to estimate the volatility function $s(x)$ of the process $(Y_i)$ based on the observations $Y_0, \ldots, Y_n$.

The conditional volatility $s_i(x)$ and the conditional variance $v_i(x)$ respectively at time $i$ is defined by

$$v_i(x) = s_i^2(x) = \mathsf{E}[Y_i^2 \,|\, Y_{i-1} = x] - \mathsf{E}^2[Y_i \,|\, Y_{i-1} = x]. \tag{14.8}$$

Included in the assumptions of the model (14.1) is the independence from the time index $i$. An estimate for $v(x) = s^2(x)$ using the LP Method is based on the fact that the two dimensional marginal distribution $(Y_{i-1}, Y_i)$ is independent of $i$. In the following we will see that $(Y_i)$ approach is a stationary process, with which the following application is justified.

Referring back to the representation (14.8) of the conditional variance $v(x)$ we search for an estimator $\hat{v}_n$ for $v$ with the form

$$\hat{v}_n(x) = \hat{g}_n(x) - \hat{f}_n^2(x), \tag{14.9}$$

i.e., we are looking for an estimator $\hat{g}_n(x)$ for $g(x) = f^2(x) + s^2(x)$ and an estimator $\hat{f}_n(x)$ for $f(x)$.

In order to define these two estimators with the LP Method, after applying the steps discussed in the previous section we have to solve both of the following minimisation problems:

$$\begin{aligned} \bar{c}_n(x) &= \arg\min_{c \in \mathbb{R}^{p+1}} \textstyle\sum_{i=1}^n (Y_i^2 - c^\top U_{in})^2 K\left(\frac{Y_{i-1}-x}{h_n}\right), \\ c_n(x) &= \arg\min_{c \in \mathbb{R}^{p+1}} \textstyle\sum_{i=1}^n (Y_i - c^\top U_{in})^2 K\left(\frac{Y_{i-1}-x}{h_n}\right). \end{aligned} \tag{14.10}$$

Here $K : \mathbb{R} \longrightarrow \mathbb{R}$ is a kernel and $\{h_n\}$ a series of positive numbers (bandwidth) with $\lim_{n\to\infty} h_n = 0$. The vectors $U_{in}$ from (14.10) are defined by

$$U_{in} = F(u_{in}), \quad u_{in} = \frac{Y_{i-1} - x}{h_n}. \tag{14.11}$$

with $\mathbb{R}^{p+1}$ valued function $F(u) = \{F_0(u), ..., F_p(u)\}^\top$ given by

$$F_k(u) = \frac{u^k}{k!}.$$

According to the LP Method we define $\hat{g}_n$ and $\hat{f}_n$ with

$$\hat{g}_n(x) = \bar{c}_n(x)^\top F(0) \quad \text{and} \quad \hat{f}_n(x) = c_n(x)^\top F(0),$$

which the above mentioned application ensures that

$$\hat{v}_n(x) = \bar{c}_n(x)^\top F(0) - \left\{c_n(x)^\top F(0)\right\}^2. \tag{14.12}$$

This estimate is a direct modification of the estimator from the local polynomial, nonparametric regression in Tsybakov (1986).

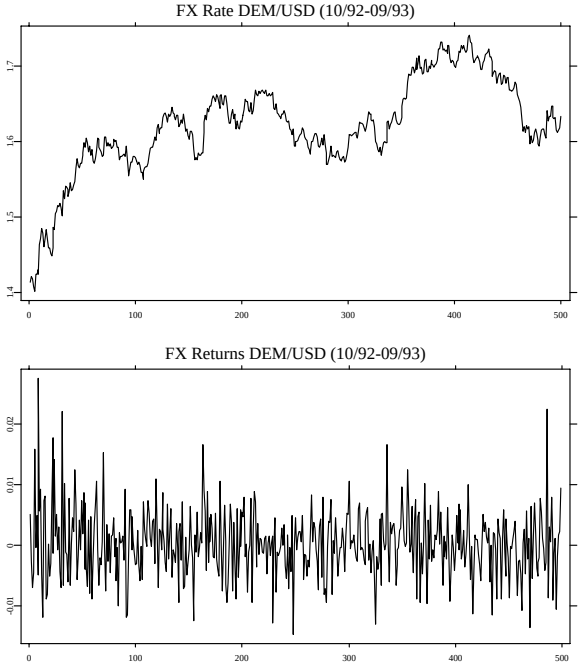


Figure 14.1: DEM/USD Exchange rate and its returns SFEdmusrate

To illustrate the estimator we consider an example. Figure 14.1 above displays the DEM/USD exchange rate from 1 October, 1992 to 30 September, 1993 in 20 minute intervals (volatility time scale). There are $n = 25434$ observations. We have calculated the returns of this series (see Figure 14.1 below) and applied the estimator (14.12) to the time series of the returns. Under the model for the geometric Brownian motion for the price, which is based on the Black-Scholes method (see Section 6.1), the returns must follow an ordinary Brownian motion. Their volatilities $v(x)$ are thus constant and independent of $x$. The estimated conditional variance functions (see Figure 14.2) show

a U shaped structure, which is called a "smiling face" structure or *smile*. The estimated volatility functions $\hat{s}(x) = \sqrt{\hat{v}(x)}$ appear to be qualitatively analogous. This means that the expected risk of the returns is significantly higher when extreme values were observed in the period previously.

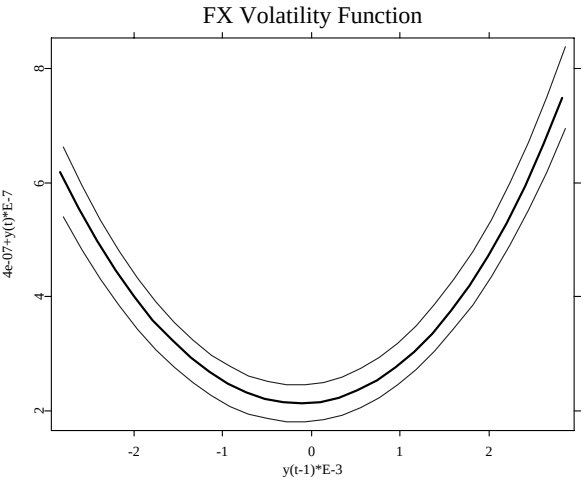


Figure 14.2: The estimated conditional variance function $\hat{v}(x)$ of the DEM/USD returns SFEdmusvol

As an alternative to equation (14.9) it is also possible to first determine the sample residuals

$$\hat{Z}_i = Y_i - \hat{f}_n(Y_{i-1}), \quad i = 1, ..., n, .$$

They approximate the true residuals $Z_i = Y_i - f(Y_{i-1})$, which under the assumptions of model (14.1) satisfy

$$E[Z_i|Y_{i-1} = x] = 0, \quad E[Z_i^2|Y_{i-1} = x] = v(x).$$

The volatility can be estimated as in the previous section directly from the nonparametric regression of $\hat{Z}_i^2$ on $Y_{i-1}$. Fan and Yao (1998) have shown that this process has advantages in heteroscedastic regression models. In estimating $f(x)$ and $v(x) = s^2(x)$, various bandwidths may be used that do not encounter the danger of the variance estimator taking on negative values. This makes sense when it is expected that the local fluctuations of $f$ and $s$ are of very different proportions.

## 14.3 Asymptotic Normality

We will now show that the estimator $\hat{v}_n(x)$ defined by (14.12) is asymptotically normally distributed. For this we will assume several technical conditions for the model. These will ensure, among other things, that the process $(Y_i)$ is ergodic. It holds that:

(A1) $\mathsf{E}[\xi_1^2] = 1$, $\mathsf{E}[\xi_1] = \mathsf{E}[\xi_1^3] = 0$, and

$$m_4 = \mathsf{E}\big[(\xi_1^2 - 1)^2\big] < \infty.$$

(A2) $\xi_1$ has a probability density $p$, that

$$\inf_{x \in \mathcal{K}} p(x) > 0$$

for every compact subset $\mathcal{K} \subset \mathbb{R}$.

(A3) There exist constants $C_1,\ C_2 > 0$, such that

$$|f(y)| \leq C_1(1 + |y|), \tag{14.13}$$

$$|s(y)| \leq C_2(1 + |y|), \quad y \in \mathbb{R}. \tag{14.14}$$

(A4) For the function $s$ it holds that

$$\inf_{y \in \mathcal{K}} s(y) > 0,$$

for every compact subset $\mathcal{K} \subset \mathbb{R}$.

(A5) $C_1 + C_2 E|\xi_1| < 1$.

With (A2) and (A4) it is certain that the process $(Y_i)$ does not die out, whereas conditions (A3) and (A5) ensure that $(Y_i)$ does not explode. These simply formed conditions can be relaxed at a large technical cost as in Franke, Kreiss, Mammen and Neumann (2003). In particular the linear growth condition (A3) must only hold asymptotically for $|y| \to \infty$.

The model (14.1) implies that $(Y_i)$ is a Markov chain. From the following lemma from Ango Nze (1992) it follows that the chain is ergodic. It is based on applications of the results given in Nummelin and Tuominen (1982) and Tweedie (1975).

**Lemma 14.1** *Under conditions (A1) - (A5) the Markov chain* $(Y_i)$ *is geometrically ergodic, i.e.,* $(Y_i)$ *is ergodic with a stationary probability density* $\pi$*, and a* $\rho \in [0, 1)$ *exists, so that for almost all* $y$ *it holds that*

$$\|P^n(\,\cdot\,|y) - \pi\|_{TV} = \mathcal{O}(\rho^n).$$

*Here*

$$P^n(B \,|\, y) = P(Y_n \in B \,|\, Y_0 = y), \quad B \in \mathcal{B},$$

*represents the conditional distribution of* $Y_n$ *given* $Y_0 = y$*, and*

$$\| \nu \|_{TV} = \sup \left\{ \sum_{i=1}^{k} |\nu(B_i)| \,;\, k \in \mathbb{N},\, B_1, \ldots, B_k \in \mathcal{B} \text{ pairwise disjunct} \right\}$$

*is the total variation of a signed measure* $\nu$ *of the Borel* $\sigma$*-Algebra* $\mathcal{B}$ *on* $\mathbb{R}$*.*

To derive the asymptotic normality from $\hat{v}_n(x)$ at a fixed point $x \in \mathbb{R}$ we require additional conditions. To simplify notation $l = p + 1$.

(A6) The functions $f$ and $s$ are at the point $x$ $(l-1)$-times continuously differentiable, and the one sided derivative $f^l_{\pm}(x)$, $s^l_{\pm}(x)$ of $l$-th order exists.

(A7) The stationary distribution $\pi$ has a bounded, continuous probability density $\gamma$, which is strictly positive in a small region around $x$.

(A8) The kernel $K : \mathbb{R} \longrightarrow \mathbb{R}^+$ is bounded with compact support and it holds that $K > 0$ for a set of positive Lebesgue measures.

(A9) The bandwidth $h_n$ is of the form $h_n = \beta n^{-1/(2l+1)}$, where $\beta > 0$.

(A10) The initial value $Y_0$ is a real number and is constant.

According to lemma 1 in Tsybakov (1986) it follows from (A8), that the matrices

$$\begin{aligned} A &= \int F(u)\, F(u)^\top\, K(u)\, du \quad \text{and} \\ Q &= \int F(u)\, F(u)^\top\, K^2(u)\, du \end{aligned}$$

are positive definite. Let

$$\mathcal{D} = A^{-1} Q A^{-1} \quad \text{and}$$

$$f^{(l)}(x;u) = \begin{cases} f_+^{(l)}(x), & u \geq 0, \\ f_-^{(l)}(x), & u < 0, \end{cases}$$

With this we define the asymptotic errors

$$\begin{aligned} b_f(x) &= A^{-1} \frac{\beta^l}{l!} \int F(u)\, u^l\, K(u)\, f^{(l)}(x;u)\, du \quad \text{and} \\ b_g(x) &= A^{-1} \frac{\beta^l}{l!} \int F(u)\, u^l\, K(u)\, g^{(l)}(x;u)\, du. \end{aligned}$$

Furthermore, let

$$c(x) = \begin{pmatrix} f(x) \\ f'(x)h_n \\ \vdots \\ f^{(l-1)}(x)\frac{h_n^{l-1}}{(l-1)!} \end{pmatrix} \quad \text{and} \quad \bar{c}(x) = \begin{pmatrix} g(x) \\ g'(x)h_n \\ \vdots \\ g^{(l-1)}(x)\frac{h_n^{l-1}}{(l-1)!} \end{pmatrix}.$$

The assertions of the following theorem is the central result of this chapter.

**Theorem 14.1** *Under assumptions (A1) - (A10) it holds that*

$$\left\{\bar{c}_n(x) - \bar{c}(x)\right\}^\top F(0) \xrightarrow{\mathrm{P}} 0, \qquad \left\{c_n(x) - c(x)\right\}^\top F(0) \xrightarrow{\mathrm{P}} 0 \tag{14.15}$$

*and*

$$n^{l/(2l+1)} \begin{pmatrix} \bar{c}_n(x) - \bar{c}(x) \\ c_n(x) - c(x) \end{pmatrix} \xrightarrow{\mathcal{L}} N\{b(x), \Sigma(x)\} \tag{14.16}$$

*for $n \to \infty$, where*

$$b(x) = \begin{pmatrix} b_g(x) \\ b_f(x) \end{pmatrix}$$

*and*

$$\Sigma(x) = \frac{s^2(x)}{\beta\gamma(x)} \begin{pmatrix} 4f^2(x) + s^2(x)m_4 & 2f(x) \\ 2f(x) & 1 \end{pmatrix} \otimes \mathcal{D}.$$

*Here $\mathcal{D}' \otimes \mathcal{D}$ represents the Kronecker product of matrices $\mathcal{D}'$ and $\mathcal{D}$.*

**Proof:**
The normal equation for the first least squares problem in (14.10) is given by

$$n^{\frac{l}{2l+1}} B_n \bar{c}_n(x) = n^{-\frac{l}{2l+1}} \sum_{i=1}^{n} Y_i^2 \, U_{in} \, K(u_{in}) \tag{14.17}$$

with the matrix

$$B_n = n^{-\frac{2l}{2l+1}} \sum_{i=1}^{n} U_{in} \, U_{in}^\top \, K(u_{in}).$$

On the other hand it holds under the definition of $B_n$

$$n^{\frac{l}{2l+1}} B_n \bar{c}(x) = n^{-\frac{l}{2l+1}} \sum_{i=1}^{n} U_{in} \, U_{in}^\top \, \bar{c}(x) \, K(u_{in}), \tag{14.18}$$

from which together with (14.17) we get

$$n^{\frac{l}{2l+1}} B_n \left\{\bar{c}_n(x) - \bar{c}(x)\right\} = n^{-\frac{l}{2l+1}} \sum_{i=1}^{n} \left\{Y_i^2 - U_{in}^\top \, \bar{c}(x)\right\} U_{in} \, K(u_{in}). \tag{14.19}$$

From the model assumptions (14.1) it follows that

$$\begin{aligned} Y_i^2 &= \Big\{ f\big(Y_{i-1}\big) + s\big(Y_{i-1}\big)\xi_i \Big\}^2 \\ &= f^2\big(Y_{i-1}\big) + 2f\big(Y_{i-1}\big)s\big(Y_{i-1}\big)\xi_i + \big(\xi_i^2 - 1\big)\, s^2\big(Y_{i-1}\big) + s^2\big(Y_{i-1}\big) \\ &= g(Y_{i-1}) + \alpha_i \end{aligned} \tag{14.20}$$

with

$$\alpha_i = 2f(Y_{i-1})\, s(Y_{i-1})\, \xi_i + s^2(Y_{i-1})(\xi_i^2 - 1).$$

According to the definition of $U_{in}$ and $\bar{c}(x)$ it holds that $U_{in}^\top \bar{c}(x) = \sum_{j=0}^{l-1} \frac{1}{j!} g^{(j)}(x)\big(Y_{i-1} - x\big)^j$. Through a Taylor expansion of $g = f^2 + s^2$ we obtain by using the integral representation of the remainder

$$\begin{aligned} g(Y_{i-1}) - U_{in}^\top \bar{c}(x) &= \frac{(Y_{i-1} - x)^l}{(l-1)!} \int_0^1 g^{(l)}\ \{x + t(Y_{i-1} - x)\}\, (1-t)^{l-1}\, dt \\ &= r_g(Y_{i-1}, x). \end{aligned} \tag{14.21}$$

From (14.19), (14.20) and (14.21) we obtain

$$\begin{aligned} & n^{\frac{l}{2l+1}} B_n\ \{\bar{c}_n(x) - \bar{c}(x)\} \\ &= n^{-\frac{l}{2l+1}} \sum_{i=1}^n \big\{ g(Y_{i-1}) - U_{in}^\top \bar{c}(x) \big\}\, U_{in}\, K(u_{in}) \\ &\quad + \ n^{-\frac{l}{2l+1}} \sum_{i=1}^n \big\{ 2f(Y_{i-1})\, s(Y_{i-1})\, \xi_i + (\xi_i^2 - 1) s^2(Y_{i-1}) \big\} U_{in}\, K(u_{in}) \\ &= \bar{b}_n(x) + \bar{q}_n(x) \end{aligned} \tag{14.22}$$

with

$$\bar{b}_n(x) = n^{-\frac{l}{2l+1}} \sum_{i=1}^n r_g(Y_{i-1}, x)\, U_{in}\, K(u_{in})$$

and

$$\bar{q}_n(x) = n^{-\frac{l}{2l+1}} \sum_{i=1}^n \alpha_i\, U_{in}\, K(u_{in}),$$

In an analogous fashion one obtains

$$n^{\frac{l}{2l+1}} B_n \big\{ c_n(x) - c(x) \big\} = b_n(x) + q_n(x) \tag{14.23}$$

with

$$b_n(x) = n^{-\frac{l}{2l+1}} \sum_{i=1}^{n} r_f(Y_{i-1}, x)\, U_{in}\, K(u_{in})$$

and

$$q_n(x) = n^{-\frac{l}{2l+1}} \sum_{i=1}^{n} \beta_i\, U_{in}\, K(u_{in}),$$

where $\beta_i = s(Y_{i-1})\xi_i$ has been substituted in.

Referring back to the representations (14.22) and (14.23) the remaining steps of the proof of Theorem 14.1 are as follows:

a) First we show that

$$B_n \xrightarrow{\mathrm{P}} B \quad \text{for } n \to \infty \tag{14.24}$$

is fulfilled for each element. Here the matrix $B = \beta\,\gamma(x)\,A$ is positive definite.

b) Next we prove the relationships

$$\bar{b}_n(x) \xrightarrow{\mathrm{P}} B\, b_g(x) \quad \text{for } n \to \infty \tag{14.25}$$

and

$$b_n(x) \xrightarrow{\mathrm{P}} B\, b_f(x) \quad \text{for } n \to \infty. \tag{14.26}$$

c) The common random vector $\left(\bar{q}_n(x), q_n(x)\right)^\top$ is asymptotically normally distributed:

$$\begin{pmatrix} \bar{q}_n(x) \\ q_n(x) \end{pmatrix} \xrightarrow{\mathcal{L}} \mathrm{N}(0, \Sigma_0) \quad \text{for } n \to \infty \tag{14.27}$$

with the covariance matrix

$$\Sigma_0 = s^2(x)\beta\gamma(x) \begin{pmatrix} 4f^2(x) + s^2(x)m_4 & 2f(x) \\ 2f(x) & 1 \end{pmatrix} \otimes Q.$$

d) It holds that

$$\begin{aligned} n^{-l/(2l+1)} q_n^\top(x)\, F(0) &\xrightarrow{\mathrm{P}} 0 \quad \text{and} \\ n^{-l/(2l+1)} \bar{q}_n^\top(x)\, F(0) &\xrightarrow{\mathrm{P}} 0 \end{aligned} \tag{14.28}$$

for $n \to \infty$.

With statements a) to d) proven, the statement of the theorem can be shown in the following way:
from b) and d) it follows that

$$\begin{aligned} & B_n \left\{ \bar{c}_n(x) - \bar{c}(x) \right\}^\top F(0) \qquad (14.29) \\ = \quad & n^{-l/(2l+1)} \bar{b}_n(x)\, F(0) + n^{-l/(2l+1)} \bar{q}_n(x)\, F(0) \xrightarrow{\mathrm{P}} 0 \end{aligned}$$

for $n \to \infty$. Because of a) and the definite results of the boundary matrix this implies that $\left\{ \bar{c}_n(x) - \bar{c}(x) \right\}^\top F(0) \xrightarrow{\mathrm{P}} 0$. Similarly one can show that $\left\{ c_n(x) - c(x) \right\}^\top F(0) \xrightarrow{\mathrm{P}} 0$.

The asymptotic Normality (14.16) can be seen in a similar way:
because of b) and c) it holds that

$$\begin{aligned} n^{\frac{l}{2l+1}} B_n \begin{pmatrix} \bar{c}_n(x) - \bar{c}(x) \\ c_n(x) - c(x) \end{pmatrix} \quad &= \quad \begin{pmatrix} \bar{b}_n(x) \\ b_n(x) \end{pmatrix} + \begin{pmatrix} \bar{q}_n(x) \\ q_n(x) \end{pmatrix} \\ &\xrightarrow{\mathcal{L}} \quad \mathrm{N}\left( \begin{pmatrix} B\, b_g(x) \\ B\, b_f(x) \end{pmatrix}, \Sigma_0 \right), \end{aligned}$$

from which, according to a) the validity of (14.16) follows.

□

a) to d) remains to be proven. To do this we need a couple of helpful results.

**Lemma 14.2** (Davydov (1973))

*Let $(Y_i)$ be a geometric ergodic Markov chain, so that $Y_0$ is distributed according to the stationary measures $\pi$ of the chain. The chain is geometric and strongly mixed, i.e., it is strongly mixing ($\alpha$-mixing) with mixing coefficients $\alpha(n)$, where $\alpha(n) \le c_0\, \rho_0^n$ for particular $0 < \rho_0 < 1$ and $c_0 > 0$ is fulfilled.*

Let $(\mathcal{F}_k)$ be the canonical filter of the process $(Y_k)$, i.e. $\mathcal{F}_k = \sigma(Y_k, Y_{k-1}, \ldots, Y_0)$ represents the generated $\sigma$-algebra from $Y_0, \ldots, Y_k$.

**Lemma 14.3** (Liptser and Shirjaev (1980), Corollary 6)

*For every $n > 0$ the series $(\eta_{nk}, \mathcal{F}_k)$ is a quadratic integrable Martingale difference, i.e.,*

$$\mathsf{E}[\eta_{nk} \,|\, \mathcal{F}_{k-1}] = 0, \quad \mathsf{E}[\eta_{nk}^2] < \infty, \quad 1 \le k \le n, \qquad (14.30)$$

*and it holds that*

$$\sum_{k=1}^{n} \mathsf{E}[\eta_{nk}^2] = 1, \quad \forall\, n \ge n_0 > 0. \qquad (14.31)$$

*Then the conditions*

$$\sum_{k=1}^{n} \mathsf{E}[\eta_{nk}^2 \,|\, \mathcal{F}_{k-1}] \xrightarrow{\mathrm{P}} 1 \quad \textit{for } n \to \infty, \tag{14.32}$$

$$\sum_{k=1}^{n} \mathsf{E}\Big[\eta_{nk}^2 \mathbf{1}(|\eta_{nk}| > \varepsilon) \,|\, \mathcal{F}_{k-1}\Big] \xrightarrow{\mathrm{P}} 0 \quad \textit{for } n \to \infty,\ \forall \varepsilon > 0 \tag{14.33}$$

*are sufficient for the distribution convergence*

$$\sum_{k=1}^{n} \eta_{nk} \xrightarrow{\mathcal{L}} N(0,1) \quad \textit{for } n \to \infty.$$

**Lemma 14.4** *Let $\phi_1$ be a continuous, bounded function and let $\phi_2$ be a bounded function. Under conditions (A1) through to (A10) it holds for every process that $Y_i, i \geq 0$, which fulfils (14.1)*

$$\begin{aligned} n^{-\frac{2l}{2l+1}} \sum_{i=1}^{n} \phi_1(Y_{i-1})\, \phi_2(u_{in})\, K(u_{in}) \\ \xrightarrow{\mathrm{P}} \beta\gamma(x)\, \phi_1(x) \int \phi_2(u)\, K(u)\, du \\ n^{-\frac{2l}{2l+1}} \sum_{i=1}^{n} \mathsf{E}\big[\phi_1(Y_{i-1})\, \phi_2(u_{in})\, K(u_{in})\big] \\ \longrightarrow \beta\gamma(x)\, \phi_1(x) \int \phi_2(u)\, K(u)\, du \end{aligned} \tag{14.34}$$

*for $n \to \infty$.*

**Proof:**
We will first prove this for the case where the Markov chain begins in equilibrium and then work our way back to the general case.

For this let $(Y_i^*)$ be a Markov chain, which fulfils (14.1) and which for $Y_0^*$ has the stationary distribution $\pi$ of $(Y_i)$ introduced in Lemma 14.1. This chain is constructed to be stationary, and by applying Lemma 14.2 we get that $(Y_i^*)$ is a geometric strong mixing process. From this it follows that

$$n^{-\frac{2l}{2l+1}} \sum_{i=1}^{n} \phi_1(Y_{i-1}^*)\, \phi_2(u_{in}^*)\, K(u_{in}^*) - n^{\frac{1}{2l+1}}\, \mathsf{E}\big[\phi_1(Y_1^*)\, \phi_2(u_{1n}^*)\, K(u_{1n}^*)\big] \xrightarrow{\mathrm{P}} 0 \tag{14.35}$$

for $n \to \infty$, where $u_{in}^* = (Y_{i-1}^* - x)/h_n$ was substituted in. For the second term in (14.35) it holds that

$$\begin{aligned} & n^{\frac{1}{2l+1}} \mathsf{E}\big[\phi_1(Y_1^*)\,\phi_2(u_{1n}^*)\,K(u_{1n}^*)\big] \\ &= \beta \frac{1}{h_n} \int \phi_1(y)\,\phi_2\left(\frac{y-x}{h_n}\right) K\left(\frac{y-x}{h_n}\right) \gamma(y)\,dy \\ &= \beta\gamma(x)\,\phi_1(x) \int \phi_2(u)\,K(u)\,du\ \{1+\mathcal{o}(1)\} \end{aligned} \tag{14.36}$$

for $n \to \infty$. Together with (14.35) it follows that for $(Y_i^*)$ (14.34) is fulfilled. Define

$$\zeta_i = \phi_1(Y_{i-1})\,\phi_2(u_{in})\,K(u_{in}), \quad \zeta_i^* = \phi_1(Y_{i-1}^*)\,\phi_2(u_{in}^*)\,K(u_{in}^*),$$

and choose a series $\{\delta_n\}$ with $\delta_n = \mathcal{o}(n^{\frac{2l}{2l+1}})$ and $\lim_{n\to\infty} \delta_n = \infty$. It follows that

$$\begin{aligned} n^{-\frac{2l}{2l+1}} \sum_{i=1}^{n} \big|\mathsf{E}[\zeta_i - \zeta_i^*]\big| &\le n^{-\frac{2l}{2l+1}} \left[ \sum_{i=1}^{\delta_n - 1} \big|\mathsf{E}[\zeta_i - \zeta_i^*]\big| + \sum_{i=\delta_n}^{n} \big|\mathsf{E}[\zeta_i - \zeta_i^*]\big| \right] \\ &\le 2n^{-\frac{2l}{2l+1}} \delta_n\, \|\phi_1\phi_2\,K\|_\infty + n^{-\frac{2l}{2l+1}} \sum_{i=\delta_n}^{n} \big|\mathsf{E}[\zeta_i - \zeta_i^*]\big| \\ &= n^{-\frac{2l}{2l+1}} \sum_{i=\delta_n}^{n} \big|\mathsf{E}[\zeta_i - \zeta_i^*]\big| + \mathcal{o}(1) \end{aligned} \tag{14.37}$$

for $n \to \infty$. From the geometric ergodicity of $(Y_i)$, according to Lemma 14.1 we obtain for the left hand side of the last expression

$$\begin{aligned} n^{-\frac{2l}{2l+1}} \sum_{i=\delta_n}^{n} \big|\mathsf{E}[\zeta_i - \zeta_i^*]\big| &= n^{-\frac{2l}{2l+1}} \sum_{i=\delta_n}^{n} \big|\mathsf{E}\big[\phi_1(Y_{i-1})\,\phi_2(u_{in})\,K(u_{in}) \\ &\qquad - \phi_1(Y_{i-1}^*)\,\phi_2(u_{in}^*)\,K(u_{in}^*)\big]\big| \\ &\le n^{-\frac{2l}{2l+1}} \sum_{i=\delta_n}^{n} \|\phi_1\phi_2\,K\|_\infty \int \big|\gamma_i(y) - \gamma(y)\big|\,dy \\ &= \mathcal{O}\left( n^{-\frac{2l}{2l+1}} \sum_{i=\delta_n}^{n} \rho^i \right) = \mathcal{o}(1) \end{aligned} \tag{14.38}$$

for $n \to \infty$, where $\gamma_i$ represents the density of $Y_{i-1}$. Thus is holds that

$$\lim_{n\to\infty} n^{-\frac{2l}{2l+1}} \sum_{i=1}^{n} \big|\mathsf{E}[\zeta_i - \zeta_i^*]\big| = 0.$$

From this it follows with the help of the Markov inequality that (14.34) also applies to $(Y_i)$. □

**Proof:**
(for Theorem 14.1, continuation)
Conditions a) to d) remains to be proven.

a) Using the definition of $B_n$ it holds for the elements of this matrix

$$(B_n)_{j,\,k} = n^{-\frac{2l}{2l+1}} \sum_{i=1}^{n} \frac{u_{in}^{k+j-2}}{(k-1)!(j-1)!}\, K(u_{in}).$$

These take on the form defined in Lemma 14.4, and it follows that:

$$(B_n)_{j,k} \xrightarrow{\mathrm{P}} \frac{\beta\,\gamma(x)}{(k-1)!(j-1)!} \int u^{k+j-2}\, K(u)\, du,$$

according to the definition of matrix $A$ this is the same as $B_n \xrightarrow{\mathrm{P}} \beta\gamma(x)A = B$. The definiteness of $A$ carries over to $B$.

b) With $f$ and $s$ fulfilled, $g = f^2 + s^2$ holds, condition (A6). For the remainder from the Taylor expansion of $g$ it holds that:

$$\begin{aligned} r_g(Y_{i-1}, x) &= u_{in}^l h_n^l \frac{1}{(l-1)!} \int_0^1 g^{(l)}\left\{x + t(Y_{i-1} - x)\right\}(1-t)^{l-1}\, dt \\ &= u_{in}^l\, n^{-\frac{l}{2l+1}}\, \phi_3(Y_{i-1}) \end{aligned}$$

with

$$\phi_3(Y_{i-1}) = \frac{\beta^l}{(l-1)!} \int_0^1 g^{(l)}\left\{x + t(Y_{i-1} - x)\right\}(1-t)^{l-1}\, dt.$$

With this $\bar{b}_n(x)$ can be rewritten as

$$\bar{b}_n(x) = n^{-\frac{2l}{2l+1}} \sum_{i=1}^{n} \phi_3(Y_{i-1})\, u_{in}^l\, U_{in}\, K(u_{in}),$$

i.e., the elements of $\bar{b}_n(x)$ fulfil the requirements of Lemma 14.4.

Once again we choose $(Y_i^*)$ as in the proof to Lemma 14.4 and set $U_{in}^* = F(u_{in}^*)$. From (14.37) and (14.38) we obtain

$$\bar{b}_n(x) - n^{-\frac{2l}{2l+1}} \sum_{i=1}^{n} \phi_3(Y_{i-1}^*)\,(u_{in}^*)^l\, U_{in}^*\, K(u_{in}^*) \xrightarrow{\mathrm{P}} 0 \qquad (14.39)$$

for $n \to \infty$. Since $(Y_i^*)$ is $\alpha$-mixing, as in (14.35) we get

$$n^{-\frac{2l}{2l+1}} \sum_{i=1}^{n} \phi_3(Y_{i-1}^*)\,(u_{in}^*)^l\,U_{in}^*\,K(u_{in}^*) - n^{\frac{1}{2l+1}}\,\mathsf{E}\Big[\phi_3(Y_1^*)\,(u_{1n}^*)^l\,U_{1n}^*\,K(u_{in}^*)\Big] \xrightarrow{\mathrm{P}} 0$$

for $n \to \infty$. The right term of this expression can be rewritten as

$$n^{\frac{1}{2l+1}}\,\mathsf{E}\Big[\phi_3(Y_1^*)\,(u_{1n}^*)^l\,U_{in}^*\,K(u_{in}^*)\Big] = \beta \int \phi_3(x+uh_n)\,u^l F(u)\,K(u)\,\gamma(x+uh_n)\,du.$$

Furthermore, it holds that

$$\lim_{n\to\infty} \phi_3(x+uh_n) = \beta^l\,g^{(l)}(x;u)/l \tag{14.40}$$

for every $u \in \mathbb{R}$. Together with (14.40) and (A7), it follows that

$$\begin{aligned}\lim_{n\to\infty} \beta \int \phi_3(x+uh_n)\,u^l F(u)\,K(u)\,\gamma(x+uh_n)\,du &= \frac{\beta^{l+1}}{l!}\left\{\int F(u)\,u^l\,K(u)\,g^{(l)}(x;u)\,du\right\}\gamma(x) \\ &= A\,\gamma(x)\,\beta\,b_g(x) = B\,b_g(x).\end{aligned}$$

With this (14.25) has been shown. The proof for (14.26) follows analogously.

c) We define the matrices

$$\begin{aligned}\Sigma_n^{11} &= n^{-\frac{2l}{2l+1}} \sum_{i=1}^{n} \mathsf{E}[\alpha_i^2 \mid \mathcal{F}_{i-1}]\,U_{in}\,U_{in}^\top\,K^2(u_{in}),\\ \Sigma_n^{12} &= n^{-\frac{2l}{2l+1}} \sum_{i=1}^{n} \mathsf{E}[\alpha_i\beta_i \mid \mathcal{F}_{i-1}]\,U_{in}\,U_{in}^\top\,K^2(u_{in}),\\ \Sigma_n^{22} &= n^{-\frac{2l}{2l+1}} \sum_{i=1}^{n} \mathsf{E}[\beta_i^2 \mid \mathcal{F}_{i-1}]\,U_{in}\,U_{in}^\top\,K^2(u_{in})\end{aligned}$$

and construct the block matrix

$$\Sigma_n = \begin{pmatrix} \Sigma_n^{11} & \Sigma_n^{12} \\ \Sigma_n^{12} & \Sigma_n^{22} \end{pmatrix}.$$

The elements of $\Sigma_n^{11}$, $\Sigma_n^{12}$ and $\Sigma_n^{22}$ fulfil the requirements of Lemma 14.4. In particular, the combined functions $\phi_1(Y_{i-1})$ that appear there are in this case given by

$$\begin{aligned} \mathsf{E}[\alpha_i^2 \,|\, \mathcal{F}_{i-1}] &= 4f^2(Y_{i-1})\, s^2(Y_{i-1}) + s^4(Y_{i-1}) m_4, \\ \mathsf{E}[\alpha_i \beta_i \,|\, \mathcal{F}_{i-1}] &= 2f(Y_{i-1})\, s^2(Y_{i-1}) \quad \text{respectively} \\ \mathsf{E}[\beta_i^2 \,|\, \mathcal{F}_{i-1}] &= s^2(Y_{i-1}), \end{aligned}$$

for which (A1) has been used. One observes that the corresponding functions $\phi_1$ are, due to (A6), continuous and restricted in a small region around $x$. Since $K$ disappears outside of a compact set, this is sufficient for Lemma 14.4. With this we obtain:

$$\Sigma_n \xrightarrow{\mathrm{P}} \Sigma_0 \quad \text{and} \quad \mathsf{E}[\Sigma_n] \longrightarrow \Sigma_0 \tag{14.41}$$

for $n \to \infty$.

To prove (14.27) it is sufficient to show, using the theorem from Cramér-Wold, that

$$a^\top \begin{pmatrix} \bar{q}_n(x) \\ q_n(x) \end{pmatrix} \xrightarrow{\mathcal{L}} \mathrm{N}(0, a^\top \Sigma_0 a) \quad \text{for } n \to \infty \tag{14.42}$$

for every vector $a \in \mathbb{R}^{2l}$ with a Euclidian norm $\|a\| = 1$ is fulfilled. In addition in accordance with (14.41) we choose a $n_0 \in \mathbb{N}$, so that $\mathsf{E}[\Sigma_n] > \frac{1}{2}\Sigma_0$ holds for all $n \geq n_0$, and substitute in for $n \geq n_0$,

$$\eta_{ni} = \frac{n^{-\frac{l}{2l+1}}}{\sqrt{a^\top \mathsf{E}[\Sigma_n] a}}\, a^\top \begin{pmatrix} \alpha_i\, U_{in} \\ \beta_i\, U_{in} \end{pmatrix} K(u_{in}).$$

Then

$$\sum_{i=1}^{n} \eta_{ni} = \frac{1}{\sqrt{a^\top \mathsf{E}[\Sigma_n] a}}\, a^\top \begin{pmatrix} \bar{q}_n(x) \\ q_n(x) \end{pmatrix},$$

and (14.42) is equivalent to

$$\sum_{k=1}^{n} \eta_{nk} \xrightarrow{\mathcal{L}} \mathrm{N}(0,1) \quad \text{for } n \to \infty. \tag{14.43}$$

We will now show that $(\eta_{nk})$ fulfills the requirements (14.30) to (14.33) from Lemma 14.3, from which (14.43) follows.

First notice that $\mathsf{E}[\alpha_i \,|\, \mathcal{F}_{i-1}] = 0$ a.s. and $\mathsf{E}[\beta_i \,|\, \mathcal{F}_{i-1}] = 0$ a.s. hold, from which (14.30) follows. Furthermore, one can easily show that

$$\sum_{k=1}^{n} \mathsf{E}[\eta_{nk}^2 \,|\, \mathcal{F}_{k-1}] = \frac{a^\top \Sigma_n a}{a^\top \mathsf{E}[\Sigma_n] a}.$$

Therefore (14.31) if fulfilled and from (14.41) we obtain (14.32).

We still have to show (14.33). For $n \geq n_0$,

$$\eta_{nk}^2 \leq \frac{n^{-\frac{2l}{2l+1}}}{a^\top \mathsf{E}[\Sigma_n] a} (a^\top Z_{nk})^2 \leq \frac{2n^{-\frac{2l}{2l+1}}}{a^\top \Sigma_0 a} (a^\top Z_{nk})^2 \leq \kappa_1 n^{-\frac{2l}{2l+1}} |Z_{nk}|^2,$$

with an appropriate constant $\kappa_1 > 0$ and

$$Z_{nk} = \begin{pmatrix} \alpha_k \, U_{kn} \\ \beta_k \, U_{kn} \end{pmatrix} K(u_{kn}).$$

Since $K$ is restricted and has compact support, and since $f$ and $s$ are locally bounded, a constant $\kappa_2 > 0$ exists, so that

$$\begin{aligned} \eta_{nk}^2 &\leq \kappa_1 n^{-\frac{2l}{2l+1}} (\alpha_k^2 + \beta_k^2) \, |U_{kn}|^2 \, K^2(u_{kn}) \\ &\leq \kappa_2 n^{-\frac{2l}{2l+1}} (1 + |\xi_k|^4) \, K(u_{kn}). \end{aligned}$$

From this it follows that

$$\begin{aligned} &\mathsf{E}[\eta_{nk}^2 \, \mathbf{1}(|\eta_{nk}| \geq \varepsilon) \,|\, \mathcal{F}_{k-1}] \\ &\quad \leq \kappa_2 n^{-\frac{2l}{2l+1}} \, K(u_{kn}) \\ &\qquad E\Big[(1 + |\xi_1|^4) \, \mathbf{1}\big(\sqrt{1 + |\xi_1|^4} \geq \varepsilon \, n^{\frac{l}{2l+1}} \, \kappa_2^{-1} \, \|K\|_\infty^{-1}\big)\Big] \\ &\quad = \kappa_2 n^{-\frac{2l}{2l+1}} \, K(u_{kn}) \cdot \mathcal{o}(1) \end{aligned}$$

for $n \to \infty$, where $\mathcal{o}(1)$ is independent of $k$. With this we have

$$\sum_{k=1}^{n} \mathsf{E}[\eta_{nk}^2 \, \mathbf{1}(|\eta_{nk}| \geq \varepsilon) \,|\, \mathcal{F}_{k-1}] \leq \mathcal{o}(1) \, \sum_{k=1}^{n} n^{-\frac{2l}{2l+1}} \, K(u_{kn}) \quad \text{for } n \to \infty. \tag{14.44}$$

According to Lemma 14.4 it holds for the last term that

$$n^{-\frac{2l}{2l+1}} \sum_{k=1}^{n} K(u_{kn}) \xrightarrow{\mathrm{P}} \beta\gamma(x) \int K(u) \, du \quad \text{for } n \to \infty. \tag{14.45}$$

From (14.44) and (14.45), (14.33) follows, i.e., the requirements of Lemma 14.3 are actually fulfilled, and thus (14.42) is also shown.

d) It is

$$\begin{aligned} n^{-l/(2l+1)} q_n^\top(x)\, F(0) &= n^{-2l/(2l+1)} \sum_{i=1}^n \beta_i\, U_{in}^\top\, F(0)\, K(u_{in}) \\ &= n^{-2l/(2l+1)} \sum_{i=1}^n \beta_i\, u_{in}\, K(u_{in}) \\ &= n^{-2l/(2l+1)} \sum_{i=1}^n \big(\beta_i - \mathsf{E}[\beta_i \,|\, \mathcal{F}_{i-1}]\big)\, u_{in}\, K(u_{in}). \end{aligned}$$

According to (A8) the kernel $K$ is bounded, and it holds that $d^* = \max\{|u| : u \in \operatorname{supp} K\} < \infty$. Thus a constant $\kappa_0 > 0$ exists, such that

$$\begin{aligned} & \mathsf{E}\big[(n^{-l/(2l+1)} q_n^\top(x)\, F(0))^2\big] \\ = \;& n^{-\frac{4l}{2l+1}}\, \mathsf{E}\Big[\big(\sum_{i=1}^n (\beta_i - \mathsf{E}[\beta_i \,|\, \mathcal{F}_{i-1}])\; u_{in} K(u_{in})\big)^2\Big] \\ \le \;& \kappa_0 n^{-\frac{4l}{2l+1}} \sum_{i=1}^n \mathsf{E}\Big[\big(\beta_i - \mathsf{E}[\beta_i \,|\, \mathcal{F}_{i-1}]\big)^2\, \mathbf{1}(|u_{in}| \le d^*)\Big]. \end{aligned}$$

If $n$ is sufficiently large, then for the last term in the last sum it holds that

$$\begin{aligned} & \mathsf{E}\Big[\big(\beta_i - \mathsf{E}[\beta_i \,|\, \mathcal{F}_{i-1}]\big)^2\, \mathbf{1}(|u_{in}| \le d^*)\Big] \\ = \;& \mathsf{E}\left[s^2(Y_{i-1})\, \xi_i^2\, \mathbf{1}\left(\frac{|Y_{i-1} - x|}{h_n} \le d^*\right)\right] \\ = \;& \mathsf{E}\left[s^2(Y_{i-1})\, \mathbf{1}\left(\frac{|Y_{i-1} - x|}{h_n} \le d^*\right)\right] \\ \le \;& \sup_{|y-x| \le h_n d^*} s^2(y) \quad < \infty. \end{aligned}$$

Thus $n^{-l/(2l+1)} q_n^\top(x)\, F(0) \xrightarrow{\mathrm{P}} 0$ is shown. Similarly it can be shown that

$$n^{-l/(2l+1)} \bar{q}_n^\top(x)\, F(0) \xrightarrow{\mathrm{P}} 0.$$

□

As a direct consequence of Theorem 14.1 we have:

**Theorem 14.2** *Under conditions (A1) through (A10) it holds that*

$$n^{l/(2l+1)} \big\{\hat{v}_n(x) - v(x)\big\} \xrightarrow{\mathcal{L}} N\big(b_v(x), \sigma_v^2(x)\big) \quad \textit{for } n \to \infty,$$

*where*

$$\begin{aligned} b_v(x) &= F^\top(0)\,(b_g(x) - 2f(x)\,b_f(x)) \quad \textit{and} \\ \sigma_v^2(x) &= \frac{s^4(x)m_4}{\beta\gamma(x)}\,F^\top(0)\,\mathcal{D}\,F(0). \end{aligned}$$

**Proof:**
From $g(x) = \bar{c}(x)^\top F(0)$, $f(x) = c(x)^\top F(0)$, $v(x) = g(x) - f^2(x)$ and the construction of $\hat{v}_n$ we obtain

$$\begin{aligned} \hat{v}_n(x) - v(x) &= \{\bar{c}_n(x) - \bar{c}(x)\}^\top\ F(0) \\ &- \Big[2c(x)^\top F(0) + \{c_n(x) - c(x)\}^\top\ F(0)\Big] \\ &\quad \Big[\{c_n(x) - c(x)\}^\top\ F(0)\Big]. \end{aligned}$$

It also holds that

$$\begin{aligned} n^{l/(2l+1)}\big(\hat{v}_n(x) - v(x)\big) = \quad & n^{l/(2l+1)}\,\Psi(x) \begin{pmatrix} \bar{c}_n(x) - \bar{c}(x) \\ c_n(x) - c(x) \end{pmatrix} \\ & + \ n^{l/(2l+1)}\big((c_n(x) - c(x))^\top F(0)\big)^2 \end{aligned} \tag{14.46}$$

with the transformations matrix

$$\Psi(x) = \begin{pmatrix} F(0) \\ -2f(x)\,F(0) \end{pmatrix}^\top.$$

According to (14.15) it holds that $\{c_n(x) - c(x)\}^\top F(0) \xrightarrow{\mathrm{P}} 0$ for $n \to \infty$, from which together with (14.16) $n^{l/(2l+1)}\big\{[c_n(x) - c(x)]^\top F(0)\big\}^2 \xrightarrow{\mathrm{P}} 0$ follows. The limiting distribution of $n^{l/(2l+1)}\big\{\hat{v}_n(x) - v(x)\big\}$ is thus given by the first term of the right side of (14.46). For this we use (14.16) that

$$n^{l/(2l+1)}\big\{\hat{v}_n(x) - v(x)\big\} \xrightarrow{\mathcal{L}} \mathrm{N}\big\{\Psi(x)b(x), \Psi(x)\Sigma(x)\Psi(x)^\top\big\}$$

for $n \to \infty$. A simple calculation gives $\Psi(x)b(x) = b_v(x)$ as well as $\Psi(x)\Sigma(x)\Psi(x)^\top = \sigma_v^2(x)$, with which the claim is shown. □

Going beyond the asymptotic normality shown in Theorem 14.2, Franke et al. (2002) have shown that bootstrap methods for nonparametric volatility estimators can also be used. They consider routine kernel estimators, i.e., the special case LP estimator with $p = 0$ in (14.4), but the results can be directly applied to the general LP estimators, see also Kreiss (2000).

To illustrate, consider the case where $l = 2$. We assume that $f$ and $s$ are twice differentiable and that the kernel $K$ satisfies the condition

$$\int K(u)\,du = 1 \qquad \text{and} \qquad K(u) = K(-u).$$

Then it holds that

$$\begin{aligned}
A &= \begin{pmatrix} 1 & 0 \\ 0 & \sigma_K^2 \end{pmatrix} \quad \text{mit } \sigma_K^2 = \int u^2\,K(u)\,du, \\
Q &= \begin{pmatrix} \int K^2(u)\,du & 0 \\ 0 & \int u^2\,K^2(u)\,du \end{pmatrix}, \\
b_f(x) &= A^{-1}\frac{\beta^2\,f''(x)}{2}\begin{pmatrix} \sigma_K^2 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_K^2\beta^2 f''(x)/2 \\ 0 \end{pmatrix}, \\
b_g(x) &= A^{-1}\frac{\beta^2\,g''(x)}{2}\begin{pmatrix} \sigma_K^2 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_K^2\beta^2 g''(x)/2 \\ 0 \end{pmatrix}, \\
\mathcal{D} &= \begin{pmatrix} \int K^2(u)\,du & 0 \\ 0 & \frac{1}{\sigma_K^4}\int u^2\,K^2(u)\,du \end{pmatrix},
\end{aligned}$$

and thus

$$b_v(x) = \frac{\sigma_K^2\beta^2}{2}\Big\{\big(f^2(x)+s^2(x)\big)'' - 2f(x)\,f''(x)\Big\} = \frac{\sigma_K^2\beta^2}{2}\Big[v''(x)+2\big\{f'(x)\big\}^2\Big]$$

and

$$\sigma_v^2(x) = \frac{s^4(x)m_4}{\beta\gamma(x)}\int K^2(u)\,du = \frac{v^2(x)m_4}{\beta\gamma(x)}\int K^2(u)\,du.$$

In particular, from the normalised quadratic errors of $\hat{v}_n$ that are calculated from the asymptotic distribution, we have

$$\begin{aligned}
\mathsf{E}\big[n^{2l/2l+1}\big(\hat{v}_n(x) - v(x)\big)^2\big] &\approx b_v^2(x) + \sigma_v^2(x) \\
&= \frac{v^2(x)m_4}{\beta\gamma(x)}\int K^2(u)\,du \\
&\quad + \frac{\sigma_K^4\beta^4}{4}\Big\{v''(x) + 2\big(f'(x)\big)^2\Big\}^2.
\end{aligned}$$

Minimizing these expressions with respect to $K$ and $\beta$ results in the Epanechnikov-Kernel

$$K(u) = K^*(u) = \frac{3}{4}\mathbf{1}(1-u^2 > 0)$$

and the following values for $\beta$:

$$\beta(K) = \left( \frac{v^2(x)\, m_4 \int K^2(u)\, du}{\gamma(x)\, \sigma_K^4 \left[ v''(x) + 2\{f'(x)\}^2 \right]^2} \right)^{1/5}.$$

With this we obtain

$$\beta^* = \beta(K^*) = \left( \frac{125\, v^2(x)\, m_4}{4\gamma(x) \left[ v''(x) + 2\{f'(x)\}^2 \right]^2} \right)^{1/5}.$$

## 14.4 Recommended Literature

The model (14.1) is thoroughly studied together with financial time series, in particular under the assumptions of the ARCH structure, in Engle (1982). Until recently academic research focused mainly on the (linear) conditional mean, or it was assumed that the conditional variance was constant or, as in the ARCH models, that it had a special form. At the beginning of the eighties this deficit in the literature was corrected by Engle (1982), and Robinson (1983; 1984) and in the statistic literature by Collomb (1984) and Vieu (1995). There have also been nonparametric and semi-parametric approximations suggested in Gregory (1989), Engle and Gonzalez-Rivera (1991). Since then the interest in the nonparametric situation discussed here, in which the form of the functions $f$ and $s$ is not identified ahead of time, has clearly grown in the economics and statistics literature, see Fan and Yao (2003).

The QTARCH models (13.32) in Gouriéroux and Monfort (1992) create a generalisation of the threshold models for the conditional mean in Tong (1983). The methods from Gouriéroux and Monfort (1992) and McKeague and Zhang (1994) are based on histogram estimations of the volatility. The works from Chen and Tsay (1993a; 1993b) concentrate on additive modelling of the mean function $f$. Additive or multiplicative structures of volatility are considered in Härdle, Lütkepohl and Chen (1997), Yang, Härdle and Nielsen (1999) and Hafner (1998). The general nonparametric ARCH model is handled in Härdle, Tsybakov and Yang (1996). Franke (1999) discusses the connection between the nonparametric AR-ARCH model and the discrete version of geometric Brownian motion which is used as a foundation for the Black-Scholes applications. Franke, Härdle and Kreiss (2003) study, in connection with a special stochastic volatility model, a nonparametric de-convolution estimator for the volatility function as the first step towards the nonparametric handling of general GARCH models.

The idea of the local polynomial estimation originates in Stone (1977), Cleveland (1979) and Katkovnik (1979; 1985), who have all used it on nonparametric regression models. Statistical properties of LP estimators by nonparametric regression models (convergence, convergence rate and pointwise asymptotic normality) are derived in Tsybakov (1986). References to more recent studies in this area can be found in Fan and Gijbels (1996).

Apart from the statistical studies of the model (14.1), the utilised theoretical probability properties of the constructed process $(Y_i)$ are also of importance. This is studied in the works of Doukhan and Ghindès (1981), Chan and Tong (1985), Mokkadem (1987), Diebolt and Guégan (1990) and Ango Nze (1992). In these articles the ergodicity, geometric ergodicity and mixture properties of the process $(Y_i)$ are derived.

# Part III

# Selected Financial Applications

# 15 Pricing Options with Flexible Volatility Estimators

Since their introduction by Engle and Bollerslev, models with autoregressive, conditional heteroscedasticity (*autoregressive conditional heteroscedasticity models* or ARCH) have been successfully applied to financial market data. Thus, it is natural to discuss option pricing models where the underlying instrument follows an ARCH process. From an empirical point of view the form of the *news impact curve*, which is defined as a function of the current volatility dependent on yesterday's returns, is the dominant factor in determining the price. It is important, for example, to know whether the news impact curve is symmetric or asymmetric. In order to avoid inaccurate pricing due to asymmetries it is necessary to use flexible volatility models. In this way EGARCH models (see Section 13.2) can be used when stock prices and volatility are correlated. This model however has a weakness in that the problem of the stationarity conditions and the asymptotic of the Quasi-Maximum-Likelihood-Estimator (QMLE) is not yet completely solved. Another Ansatz, as in the Threshold GARCH-Models, is to introduce thresholds in the news impact curve to create flexible asymmetry.

In this chapter we concentrate on the specification of the volatility. We present the TGARCH process and perform Monte Carlo simulations for three typical parameter groups. In particular we compare the simulated GARCH option prices with option prices based on the simulations from TGARCH and Black-Scholes models. In the empirical section of the chapter we show that the market price of call options indeed reflect the asymmetries that were discovered in the news impact curve of the DAX time series.

## 15.1 Pricing Options with ARCH-Models

Consider an economy in discrete time in which interest and proceeds are paid out at the end of every constant, equally long time interval. Let $S_t, t = 0, 1, 2, \ldots$ be the price of the stock at time $t$ and $Y_t = (S_t - S_{t-1})/S_{t-1}$ the corresponding one period return without dividends. Assume that a price for

risk exists in the form of a risk premium which is added to the risk free interest rate $r$ to obtain the expected return of the next period. It seems reasonable to model the risk premium dependent on the conditional variance. As a basis we assume an ARCH-M-Model (see Section 13.2.3) with a risk premium, which is a linear function of the conditional standard deviation:

$$\begin{aligned} Y_t &= r + \lambda\sigma_t + \varepsilon_t && (15.1)\\ \mathcal{L}(\varepsilon_t \mid \mathcal{F}_{t-1}) &= \mathrm{N}(0, \sigma_t^2) && (15.2)\\ \sigma_t^2 &= \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2. && (15.3) \end{aligned}$$

In (15.3) $\omega$, $\alpha$ and $\beta$ are constant parameters that satisfy the stationarity and non-negativity conditions. The constant parameter $\lambda$ can be understood as the price of one unit of risk. $\mathcal{F}_t$ indicates, as usual, the set of information available up to and including time $t$. In order to simplify the notation, our discussion will be limited to the GARCH(1,1) case.

The above model is estimated under the empirical measure $P$. In order to deal with a valuation under no arbitrage, similar to Black-Scholes in continuous time (see Section 6.1), assumptions on the valuation of risk must be made. Many studies have researched option pricing with stochastic volatility under the assumption that the volatility has a systematic risk of zero, that is, the risk premium for volatility is zero. Duan (1995) has identified an equivalent martingale measure $Q$ for $P$ under the assumption that the conditional distribution of the returns are normal, and in addition it holds that

$$\mathrm{Var}^P(Y_t \mid \mathcal{F}_{t-1}) = \mathrm{Var}^Q(Y_t \mid \mathcal{F}_{t-1}) \tag{15.4}$$

$P$ a.s.. He shows that under this assumption a representative agent with, for example, constant relative risk aversion and a normally distributed relative change of aggregate consumption maximises his expected utility. The assumption (15.4) contains a constant risk premium for the volatility that directly enters its mean.

In order to obtain a martingale under the new measure a new error term, $\eta_t$, needs to be introduced that captures the effect of the time varying risk premium. When we define $\eta_t = \varepsilon_t + \lambda\sigma_t$, (15.4) it leads to the following model under the new measure $Q$:

$$\begin{aligned} Y_t &= r + \eta_t && (15.5)\\ \mathcal{L}_Q(\eta_t \mid \mathcal{F}_{t-1}) &= \mathrm{N}(0, \sigma_t^2) && (15.6)\\ \sigma_t^2 &= \omega + \alpha(\eta_{t-1} - \lambda\sigma_{t-1})^2 + \beta\sigma_{t-1}^2. && (15.7) \end{aligned}$$

In the case of a GARCH(1,1) model according to Theorem 13.10 the variance of the stationary distribution under the empirical measure $P$ is $\mathrm{Var}^P(\varepsilon_t) =$

$\omega/(1-\alpha-\beta)$. For the Duan measure $Q$ the variance of the stationary distribution increases to $\mathrm{Var}^Q(\eta_t) = \omega/\{1-\alpha(1+\lambda^2)-\beta\}$, because the volatility process under the new measure is determined by the innovations from an asymmetric and a non-symmetric Chi squared distribution. Later on we will see that changes in the unconditional variance depend, in a critical way, on the specification of the news impact curve.

The restriction to a quadratic or symmetric news impact curve is not always optimal, as many empirical studies of stock returns have indicated. Within the framework of the above mentioned model these assumptions can lead to a non-linear news impact function $g(\cdot)$. The following model is a semi-parametric analogue to the GARCH model. Under the empirical measure $P$ we obtain

$$\begin{aligned} Y_t &= r + \lambda\sigma_t + \varepsilon_t \\ \mathcal{L}_P(\varepsilon_t \mid \mathcal{F}_{t-1}) &= \mathrm{N}(0, \sigma_t^2) \\ \sigma_t^2 &= g(\varepsilon_{t-1}) + \beta\sigma_{t-1}^2. \end{aligned}$$

Under the Duan martingale measure $Q$ the model changes to

$$\begin{aligned} Y_t &= r + \eta_t \\ \mathcal{L}_Q(\eta_t \mid \mathcal{F}_{t-1}) &= \mathrm{N}(0, \sigma_t^2) \\ \sigma_t^2 &= g(\eta_{t-1} - \lambda\sigma_{t-1}) + \beta\sigma_{t-1}^2. \end{aligned}$$

One notices that as soon as an estimator of $g(\cdot)$ under $P$ is known it can immediately be substituted under the measure $Q$.

In this general specification the estimation without additional information on $g(\cdot)$ is a difficult matter, since iterative estimation procedures would be necessary in order to estimate the parameters $\lambda, \beta$ and the non-parametric function $g$ at the same time. Therefore we will consider a specific, flexible parametric model: the Threshold GARCH Model, see Section 13.2. With this model the news impact function can be written as:

$$g(x) = \omega + \alpha_1 x^2 \mathbf{1}(x < 0) + \alpha_2 x^2 \mathbf{1}(x \geq 0)$$

To motivate this model consider fitting a very simple non-parametric model $Y_t = \sigma(Y_{t-1})\xi_t$ to the returns of a German stock index, the DAX, where $\xi_t$ is independent and identically distributed with mean 0 and variance 1. The estimator of the news impact curve $\sigma^2(\cdot)$ is given in Figure 15.2. To get an idea of the underlying distribution of the returns a non-parametric estimator of the return distribution has been added in Figure 15.1 over a smoothed normal distribution. Obviously $g(\cdot)$ is not symmetric around zero. The TGARCH model captures this phenomenon when $\alpha_1 > \alpha_2$. Other parametric models

can describe these properties as well but the TGARCH model in the case of stock returns has proven to be extremely flexible and technically manageable as claimed, for example, in Rabemananjara and Zakoian (1993).

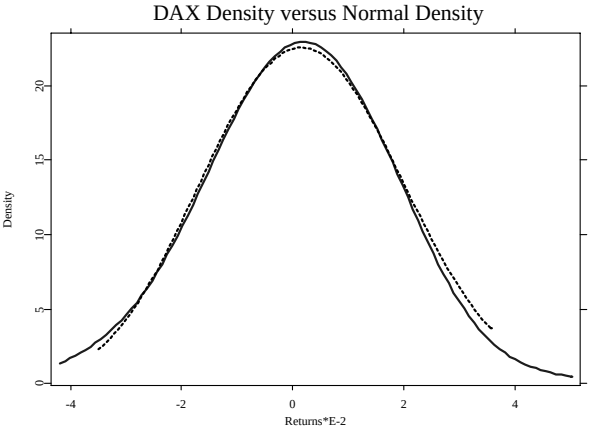


Figure 15.1: Kernel estimation of the density of DAX returns (solid line) against a kernel estimation of a normal distribution (dotted line) with the same mean and variance. A bandwidth of 0.03 is used and a quadratic kernel function $K(u) = 15/16(1-u^2)^2\mathbf{1}(|u| < 1)$. The tails have been eliminated from the figure.
SFEDaxReturnDistribution

Remember that the innovations are normally distributed. Thus it follows for the TGARCH model that the unconditional variance, similar to Theorem 13.10, under the measure $P$ is $\mathrm{Var}^P(\varepsilon_t) = \omega/(1-\bar{\alpha}-\beta)$, where $\bar{\alpha} = (\alpha_1 + \alpha_2)/2$. The following theorem gives the unconditional variance for $\eta_t = \varepsilon_t + \lambda\sigma_t$ under $Q$.

**Theorem 15.1** *The unconditional variance of the TGARCH(1,1) model under the equivalent martingale measure Q from Duan is*

$$Var^Q(\eta_t) = \frac{\omega}{1-\psi(\lambda)(\alpha_1-\alpha_2)-\alpha_2(1+\lambda^2)-\beta} \tag{15.8}$$

*where*

$$\psi(u) = u\varphi(u) + (1+u^2)\Phi(u)$$

*and $\varphi(u), \Phi(u)$ are the density and the distribution function of the standard normal distribution.*

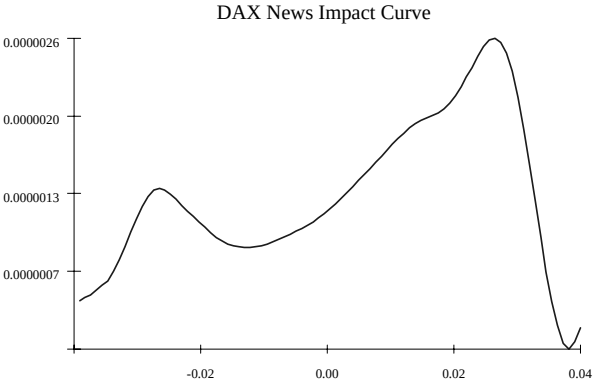


Figure 15.2: Local linear estimation of the news impact curve for the DAX. The model is $Y_t = \sigma(Y_{t-1})\xi_t$. The estimator of the function $\sigma^2(y)$ with a bandwidth of 0.03 is shown. The tails have been eliminated from the figure.

SFENewsImpactCurve

**Proof:**
Let $Z_t = \eta_t/\sigma_t - \lambda$. Under measure $Q$ it holds that $\mathcal{L}(Z_t \mid \mathcal{F}_{t-1}) = \mathrm{N}(-\lambda, 1)$. The conditional variance $\sigma_t^2$ can be written as

$$\sigma_t^2 = \omega + \alpha_1\sigma_{t-1}^2 Z_{t-1}^2 \mathbf{1}(Z_{t-1} < 0) + \alpha_2\sigma_{t-1}^2 Z_{t-1}^2 \mathbf{1}(Z_{t-1} \geq 0) + \beta\sigma_{t-1}^2.$$

By calculating the expected value it can be shown that for the integral over the negative values it follows that:

$$\begin{aligned}
\mathsf{E}^Q[Z_t^2 \mathbf{1}(Z_t < 0) \mid \mathcal{F}_{t-1}] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} z^2 e^{-\frac{1}{2}(z+\lambda)^2} dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} (u-\lambda)^2 e^{-\frac{1}{2}u^2} du \\
&= \frac{\lambda}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda^2} + (1+\lambda^2)\Phi(\lambda) \\
&\stackrel{\text{def}}{=} \psi(\lambda). \qquad (15.9)
\end{aligned}$$

Because of

$$\mathsf{E}^Q[Z_t^2 \mid \mathcal{F}_{t-1}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}(z+\lambda)^2} dz = 1 + \lambda^2$$

it follows for the positive values that

$$\mathsf{E}^Q[Z_t^2\mathbf{1}(Z_t \geq 0) \mid \mathcal{F}_{t-1}] = 1 + \lambda^2 - \psi(\lambda). \tag{15.10}$$

Thus we obtain

$$\mathsf{E}^Q[\sigma_t^2] = \omega + \alpha_1\psi(\lambda)\mathsf{E}^Q[\sigma_{t-1}^2] + \alpha_2[1 + \lambda^2 - \psi(\lambda)]\mathsf{E}^Q[\sigma_{t-1}^2] + \beta\mathsf{E}^Q[\sigma_{t-1}^2]. \tag{15.11}$$

Since the unconditional variance is independent of $t$, the theorem follows. □

The function $\psi$ is positive and $\psi(\lambda) > 1/2$ for the realistic case $\lambda > 0$. We can make the following statement about the changes in the unconditional variance: for $\alpha_1 = \alpha_2$ in (15.8), one obtains the GARCH(1,1) results. For $\alpha_1 > \alpha_2$ (the case of the leverage effect) the increase in the unconditional variance is even stronger than the symmetric GARCH case. For $\alpha_1 < \alpha_2$, the unconditional variance is smaller as in the leverage case, and we can distinguish between two cases: when the inequality

$$\alpha_1 < \alpha_2 \frac{2\psi(\lambda) - 1 - 2\lambda^2}{2\psi(\lambda) - 1} \tag{15.12}$$

is fulfilled then the unconditional variance under $Q$ is actually smaller than under $P$. If (15.12) is not fulfilled, then we obtain, as above, $\mathrm{Var}^P(\varepsilon_t) \leq \mathrm{Var}^Q(\eta_t)$. Indeed the quotient on the right hand side of (15.12) takes on negative values for realistic values of a unit of the risk premium, (for example for small positive values), so that in most empirical studies (15.12) can not be fulfilled.

Naturally the stationary variance has an effect on an option's price: the larger (smaller) the variance is, the higher (lower) the option price is. This holds in particular for options with a longer time to maturity where the long-run average of the volatility is the most important determinant of the option's price. Therefore, an option can be undervalued when a GARCH model is used and at the same time a leverage effect is present.

A second feature of the Duan approach is that under $Q$ and with positive risk premia, the current innovation is negatively correlated with the next period's conditional variance of the GARCH risk premium, whereas under $P$ the correlation is zero. More precisely, we obtain $\mathrm{Cov}^Q(\eta_t/\sigma_t, \sigma_{t+1}^2) = -2\lambda\alpha\mathrm{Var}^Q(\eta_t)$ with the GARCH parameter $\alpha$. It is obvious that small forecasts of the volatility under $Q$ (that influences the option's price) depend not only on the past squared innovations, but also on their sign. In particular a negative (positive) past innovation for $\lambda > 0$ leads to the fact that the volatility increases

(falls) and with it, the option price. The following theorem claims that the covariance is dependent on the asymmetry of the news impact function when a TGARCH instead of a GARCH model is used.

**Theorem 15.2** *For the TGARCH(1,1) model the covariance between the innovation in $t$ and the conditional variance in $t+1$ under the equivalent martingale measure $Q$ from Duan is given by*

$$Cov^Q(\frac{\eta_t}{\sigma_t}, \sigma_{t+1}^2) = -2\,Var^Q(\eta_t)[\lambda\alpha_2 + \{\varphi(\lambda) + \lambda\Phi(\lambda)\}(\alpha_1 - \alpha_2)], \quad (15.13)$$

*where $Var^Q(\eta_t)$ follows from the previous theorem.*

**Proof:**
First the conditional covariance is deterimined:

$$\begin{aligned}
\mathrm{Cov}_{t-1}^Q(\frac{\eta_t}{\sigma_t}, \sigma_{t+1}^2) &= \mathsf{E}_{t-1}^Q\left[\frac{\eta_t}{\sigma_t}\sigma_{t+1}^2\right] = \omega\mathsf{E}_{t-1}^Q\left[\frac{\eta_t}{\sigma_t}\right] \\
&+ \alpha_1\mathsf{E}_{t-1}^Q\left[\frac{\eta_t}{\sigma_t}(\eta_t - \lambda\sigma_t)^2\mathbf{1}(\eta_t - \lambda\sigma_t < 0)\right] \\
&+ \alpha_2\mathsf{E}_{t-1}^Q\left[\frac{\eta_t}{\sigma_t}(\eta_t - \lambda\sigma_t)^2\mathbf{1}(\eta_t - \lambda\sigma_t \geq 0)\right] \\
&+ \beta\sigma_t\mathsf{E}_{t-1}^Q[\eta_t], \qquad (15.14)
\end{aligned}$$

where $\mathsf{E}_t(\cdot)$ and $\mathrm{Cov}_t(\cdot)$ are abbreviations of $\mathsf{E}(\cdot \mid \mathcal{F}_t)$ and $\mathrm{Cov}(\cdot \mid \mathcal{F}_t)$ respectively. Due to (15.6) the first and the fourth expectation values on the right side of (15.14) are zero. The second conditional expected value is

$$\begin{aligned}
&\mathsf{E}_{t-1}^Q\left[\tfrac{\eta_t}{\sigma_t}(\eta_t - \lambda\sigma_t)^2\mathbf{1}(\eta_t - \lambda\sigma_t < 0)\right] \\
= \quad &-2\sigma_t^2\left[\tfrac{1}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}\lambda^2) + \lambda\Phi(\lambda)\right]. \qquad (15.15)
\end{aligned}$$

Since $\mathsf{E}_{t-1}^Q\left[\frac{\eta_t}{\sigma_t}(\eta_t - \lambda\sigma_t)^2\right] = -2\lambda\sigma_t^2$, we can write for the third conditional expected value in (15.14):

$$\begin{aligned}
&\mathsf{E}_{t-1}^Q\left[\tfrac{\eta_t}{\sigma_t}(\eta_t - \lambda\sigma_t)^2\mathbf{1}(\eta_t - \lambda\sigma_t \geq 0)\right] \\
= \quad &-2\sigma_t^2\left[\lambda - \tfrac{1}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}\lambda^2) - \lambda\Phi(\lambda)\right]. \qquad (15.16)
\end{aligned}$$

Inserting (15.15) and (15.16) into (15.14), it follows that

$$\mathrm{Cov}_{t-1}^Q(\frac{\eta_t}{\sigma_t}, \sigma_{t+1}^2) = -2\sigma_t^2[\lambda\alpha_2 + \{\varphi(\lambda) + \lambda\Phi(\lambda)\}(\alpha_1 - \alpha_2)]. \quad (15.17)$$

One notices that $\mathrm{Cov}^Q(\eta_t/\sigma_t, \sigma_{t+1}^2) = \mathsf{E}^Q[\mathrm{Cov}_{t-1}^Q(\eta_t/\sigma_t, \sigma_{t+1}^2)]$, thus the claim follows immediately. □

In the following we assume that a positive risk premium $\lambda$ exists per unit. Three cases can be identified: for $\alpha_1 = \alpha_2$ (in the symmetric case) we obtain $\mathrm{Cov}^Q(\eta_t/\sigma_t, \sigma_{t+1}^2) = -2\lambda\alpha_2\mathrm{Var}^Q(\eta_t)$, i.e., the GARCH(1,1) result. For $\alpha_1 < \alpha_2$ (the case of the reverse leverage effect) the covariance increases, and when

$$\lambda\alpha_2 + \left[\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}\lambda^2) + \lambda\Phi(\lambda)\right](\alpha_1 - \alpha_2) < 0, \tag{15.18}$$

the correlation is positive. In the last case, $\alpha_1 > \alpha_2$ (the leverage case), the covariance is negative and increases with the total.

This also shows that the return of the volatility to a stationary variance under $Q$ is different from the symmetric GARCH case. The negative covariance in the leverage case is actually larger. This could indicate that options are over (under) valued when for positive (negative) past innovation a TGARCH process with $\alpha_1 > \alpha_2$ is used for the price process and then mistakenly a GARCH model ($\alpha_1 = \alpha_2$) is used for the volatility forecast.

## 15.2 A Monte Carlo Study

Since the discounted price process is a martingale under the equivalent martingale measure $Q$, we can utilise the method of risk neutral valuation according to Cox and Ross (1976). The $Q$ price, $C_t$, of a call at time $t$ is given by the discounted conditional expectation of the payments due at maturity, see (6.23)

$$C_t = (1+r)^{-\tau}\mathsf{E}^Q[\max(S_T - K, 0) \mid \mathcal{F}_t] \tag{15.19}$$

where $T$ is the maturity date, $\tau = T - t$ is the time to maturity and $K$ is the strike price. For European options the arbitrage free price $P_t$ of a put follows from the Put-Call-Parity (Theorem 2.3), i.e., $P_t = C_t - S_t + (1+r)^{-\tau}K$. Since there is no analytical expression in a GARCH or TGARCH model for the expectation in (15.19), we have to calculate the option price numerically. The distribution of the payment function $\max(S_T - K, 0)$ at maturity is simulated in that $m$ stock processes

$$S_{T,i} = S_t \prod_{s=t+1}^{T}(1 + Y_{s,i}), \quad i = 1, \ldots, m, \tag{15.20}$$

are generated, where $Y_{s,i}$ is the return of the $i-$th replication at time $s$. Finally the mean of the payment function is discounted by the risk free interest

| Type | $\alpha$ | $\beta$ | $\alpha+\beta$ | $\rho_1$ |
|---|---|---|---|---|
| 1 | 0.1 | 0.85 | 0.95 | 0.1791 |
| 2 | 0.5 | 0.45 | 0.95 | 0.8237 |
| 3 | 0.1 | 0.5 | 0.6 | 0.1077 |

Table 15.1: Characterisation of the types of GARCH(1,1) models

rate

$$C_t = (1+r)^{-\tau}\frac{1}{m}\sum_{i=1}^{m}\max(S_{T,i}-K,0). \tag{15.21}$$

In the simulation study we used the following parameters: $r=0$, $S_0=100$, $\tau=30$ days, $m=400\,000$, $\lambda=0.01$. The *Moneyness* $S_0/K$ varies between 0.85 and 1.15, which corresponds to the usual bandwidth of the traded option. We are not comparing here the effect of various maturity dates, $T$, since many characteristics such as the smile in the stochastic volatility disappear with increasing time periods. In general the effects remain qualitatively equal, but, from a quantitative point of view, become less important. This has been shown in numerous experiments; thus we will concentrate on shorter time periods.

The effect of an asymmetric news impact function on the price of an option is studied in three different situations which are characterised by the degree of the short-run autocorrelation of the squared returns and the persistence, i.e., the value from $\alpha+\beta$. For the GARCH(1,1) process it can be shown that the autocorrelation $\rho_1$ of first order of the squared residuals is given by

$$\rho_1 = \alpha(1-\alpha\beta-\beta^2)/(1-2\alpha\beta-\beta^2), \tag{15.22}$$

and $\rho_j = (\alpha+\beta)\rho_{j-1}$, $j\geq 2$. These are the autocorrelations of an ARMA(1,1) process, since the quadratic GARCH(1,1) process satisfy a ARMA(1,1) model (see Theorem 13.9). Table 15.1 lists the parameter groups and characteristics of the three types.

Type 1 is characterised by a high persistence and a low first order correlation; type 2 is characterised by a high persistence and a high first order autocorrelation and type 3 has a low persistence and a small first order autocorrelation. Type 1 is typical for financial time series (for daily as well as intra day data), since one usually observes that the autocorrelation function of the squared returns diminishes quickly in the first few lags and then slowly after that. Type 2 describes a situation with a very strong ARCH effect, and type 3 is similar to the behaviour of heavily aggregated data such as monthly

| | | GARCH | | TGARCH | | | |
|---|---|---|---|---|---|---|---|
| | | | | Leverage Effect | | Inv. Lev. Eff. | |
| Type | Moneyness | % diff | SE | % diff | SE | % diff | SE |
| | 0.85 | 35.947 | 1.697 | 0.746 | 1.359 | 75.769 | 2.069 |
| | 0.90 | -0.550 | 0.563 | -12.779 | 0.498 | 11.606 | 0.631 |
| | 0.95 | -6.302 | 0.261 | -9.786 | 0.245 | -3.153 | 0.278 |
| Type 1 | 1.00 | -3.850 | 0.132 | -4.061 | 0.125 | -3.806 | 0.139 |
| | 1.05 | -1.138 | 0.057 | -0.651 | 0.052 | -1.692 | 0.061 |
| | 1.10 | -0.020 | 0.025 | 0.347 | 0.022 | -0.400 | 0.028 |
| | 1.15 | 0.162 | 0.012 | 0.347 | 0.010 | -0.013 | 0.014 |
| | 0.85 | 199.068 | 5.847 | 104.619 | 4.433 | 293.704 | 7.884 |
| | 0.90 | 0.489 | 1.136 | -23.964 | 0.891 | 22.140 | 1.469 |
| | 0.95 | -30.759 | 0.370 | -39.316 | 0.305 | -24.518 | 0.454 |
| Type 2 | 1.00 | -20.975 | 0.167 | -22.362 | 0.141 | -20.804 | 0.198 |
| | 1.05 | -6.038 | 0.077 | -5.427 | 0.063 | -7.148 | 0.095 |
| | 1.10 | -0.302 | 0.042 | 0.202 | 0.033 | -0.966 | 0.054 |
| | 1.15 | 0.695 | 0.027 | 0.991 | 0.021 | 0.351 | 0.037 |
| | 0.85 | -2.899 | 1.209 | -11.898 | 1.125 | 6.687 | 1.297 |
| | 0.90 | -5.439 | 0.496 | -8.886 | 0.479 | -1.982 | 0.513 |
| | 0.95 | -4.027 | 0.249 | -4.970 | 0.245 | -3.114 | 0.254 |
| Type 3 | 1.00 | -2.042 | 0.128 | -2.077 | 0.126 | -2.025 | 0.130 |
| | 1.05 | -0.710 | 0.055 | -0.559 | 0.053 | -0.867 | 0.056 |
| | 1.10 | -0.157 | 0.023 | -0.047 | 0.022 | -0.267 | 0.023 |
| | 1.15 | -0.009 | 0.010 | 0.042 | 0.010 | -0.059 | 0.011 |

Table 15.2: Simulation results for selected values of moneyness. Shown are the proportional differences between the GARCH and TGARCH option prices and the Black-Scholes price and the corresponding standard error (SE) of the simulation.

or quarterly. In every case the parameter $\omega$ is set so that $\sigma^2 = 0.0002$, i.e. the unconditional variance remains constant.

In view of the non-linear news impact function $g(\cdot)$ we choose the Threshold ARCH model with two asymmetrical cases. In the first case, which we call the leverage case,

$$g_1(x) = \omega + 1.2\alpha x^2 \mathbf{1}(x < 0) + 0.8\alpha x^2 \mathbf{1}(x \geq 0)$$

and in the second case, that of the inverse leverage effect,

$$g_2(x) = \omega + 0.8\alpha x^2 \mathbf{1}(x < 0) + 1.2\alpha x^2 \mathbf{1}(x \geq 0).$$

For type 1 and the leverage effect case the simulation results are given in Figure 15.3. We have removed the absolute and the relative difference of the GARCH and the TGARCH prices from the corresponding Black-Scholes price. The relative difference is defined as the absolute difference divided by the Black-Scholes price. Because of the small step length (we assume a step length of 0.01 for moneyness) the functions appear quite smooth. For the GARCH case we obtain the well known result that the price difference to the Black-Scholes displays a U-shape with respect to moneyness. Due to the monotone increase in moneyness of the call price, the relative difference is the largest for options out of the money. The relative difference becomes insignificantly smaller, the more it is in the money. This could explain the frequently observed skewness of the smile effect. For the TGARCH option price we observe in principle a similar deviation from Black-Scholes, although with an important difference: in the case of the leverage effect the price of the out of the money options is lower and the price of those in the money is higher than in the GARCH model. This is also plausible: when an option is way out of the money and the maturity date is close, the only way to achieve a positive payment at maturity is when the price of the underlying instrument consecutively increases in value in large jumps. This is, however, less likely in the leverage case, since positive returns have a smaller effect on the volatility than they do in the symmetric case, assuming that the parameter groups named above hold.

Table 15.2 shows the results for the type 2 and 3 and the case of the inverse leverage effect and for chosen values of moneyness. For the leverage effect case the described deviation of the TGARCH price from each GARCH price is visible even for type 2 and 3. In the case of the inverse leverage effect the arguments are reverse: it is more probable that an out of the money option can still end up in the money so that TGARCH prices of out of the money options are higher than the GARCH prices. As one would expect, the deviations of the simulated GARCH and TGARCH prices from the Black-Schole prices are the largest for type 2, i.e., for strong short-run ARCH effects, and are the smallest for the type with the lowest persistence, type 3. This last statement is to be expected, since the differences should disappear the closer we get to the homoscedastic case.

## 15.3 Application to the Valuation of DAX Calls

The valuation method with GARCH is applied to the German stock index and options data. For the stock index we use the daily closing values of the DAX from 1 January, 1988 to 31 March, 1992. The closing values are usually set

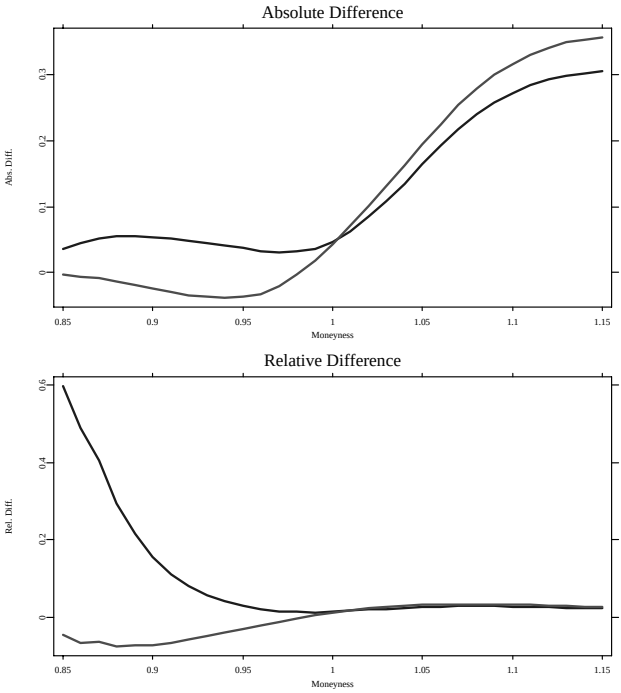


Figure 15.3: The difference between the simulated GARCH (solid line) and TGARCH (dotted line) option prices from the BS prices is given as a function of the moneyness for type 1 and the leverage case. The figure above shows the absolute differences, the figure below shows the absolute differences divided by the BS price.
SFEoptpricesim

at 13:30 (Frankfurt time). For the options data on this index we have taken the recorded values of the transaction prices from the German derivative exchange (DTB) from January to March 1992. In order to synchronise the observation time periods of the index and options we interpolate between the last option price before 13:30 and the first price after, as long as the difference is no more than two hours.

No evidence for autocorrelated DAX returns was found but the squared and absolute returns are highly autocorrelated. We estimate a GARCH(1,1)-M

| | GARCH | TGARCH |
|---|---|---|
| $\omega$ | 1.66E-05 (1.04E-06) | 1.91E-05 (1.359E-06) |
| $\alpha$ | 0.144 (0.006) | |
| $\alpha_1$ | | 0.201 (0.008) |
| $\alpha_2$ | | 0.045 (0.011) |
| $\beta$ | 0.776 (0.012) | 0.774 (0.016) |
| $\lambda$ | 0.069 (0.018) | 0.039 (0.018) |
| $-2logL$ | -7698 | -7719 |

Table 15.3: The GARCH and TGARCH estimation results for DAX returns, 1 January, 1988 to 30 December, 1991 (QMLE standard error in parentheses)

model

$$\begin{aligned} Y_t &= \lambda\sigma_t + \varepsilon_t && (15.23)\\ \mathcal{L}(\varepsilon_t \mid \mathcal{F}_{t-1}) &= \mathrm{N}(0, \sigma_t^2) && (15.24)\\ \sigma_t^2 &= \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 && (15.25) \end{aligned}$$

for the DAX with the Quasi-Maximum-Likelihood-Method - see Section 13.1.6. A possible constant in (15.23) is not significant and is thus ignored from the very beginning. Table 15.3 shows the results of the estimation. All parameters are significantly different from zero. The degree of persistence $\alpha + \beta = 0.9194$ is significantly smaller than 1 and thus the unconditional variance is finite, see Theorem 13.10. The parameter of the risk premium $\lambda$ is positive, as is expected from economic theory.

The Quasi-Maximum-Likelihood-Estimator of the TGARCH model

$$\sigma_t^2 = \omega + \alpha_1\varepsilon_{t-1}^2\mathbf{1}(\varepsilon_{t-1} < 0) + \alpha_2\varepsilon_{t-1}^2\mathbf{1}(\varepsilon_{t-1} \geq 0) + \beta\sigma_{t-1}^2 \qquad (15.26)$$

is also given in Table 15.3. Taking the value of the log-likelihood into consideration, the ability of the TGARCH model is better than that of GARCH model. A likelihood quotient test rejects the GARCH model at every rational confidence level. $\alpha_1$ and $\alpha_2$ are significantly different; thus the asymmetry of the news impact function is significant. Since $\alpha_1 > \alpha_2$ we observe the usual leverage effect for financial time series.

After the model was fitted to the data from 1988 to 1991, the next step in calculating the option price for the observed time period from January to March 1992 is to use the simulation method described above and then compare this to the market prices. Here we will concentrate on call options.

Since the DAX option, which is traded on the DTB, is a European option, the results for put options can be calculated as usual from the put-call-parity. We consider nine call options with maturity dates 17 January, 20 March, and 19 June, 1992. In order to distinguish the case of in, out and at the money, we have chosen the strike prices 1550, 1600 and 1650 for the January option 1600, 1650 and 1700 for the March and June options. We simulate the price of the January option from January 3rd to the 16th (10 days), for the March option from January 3rd to the 19th (57 days) and for the June option from January 3rd to the 31st of March (64 days). The June option with a strike price of 1700 began on 16 January so that there are no observations for the first 10 trading days. Due to low trading volume not all market prices are available thus we reduced the number of observations, $k$ in Table 15.4, even further.

A remaining question is how to choose the starting value of the volatility process. We set the starting value equal to the running estimator of the volatility (GARCH or TGARCH), in which the volatility process is extrapolated and the parameters are held constant. Alternatively one can use the implied volatility, see Section 6.4.5.

To calculate the Black-Scholes price at time $t$ the implied volatility at time $t-1$ is used. To obtain a measure of the quality of the estimate, we define the relative residuals as

$$u_t \stackrel{\text{def}}{=} \frac{C_t - C_{Market,t}}{C_{Market,t}}$$

where $C_t$ is either the Black-Scholes or the GARCH or the TGARCH price and $C_{Market,t}$ is the price observed on the market. Residuals should be considered as relative values, since a trader would always prefer the cheaper option, which is undervalued by the same amount as a more expensive option, simply because he can multiply his position in the cheaper option. A similar argument holds for the sale of an overvalued option. For reasons of symmetry we use a squared loss criterion, i.e.,

$$U = \sum_{t=1}^{k} u_t^2.$$

The results for the three models are given in Table 15.4.

Overall the GARCH as well as the TGARCH options valuation model performs substantially better than the Black-Scholes model. For options in and at the money the improvement of the TGARCH forecast compared to the GARCH model is small. When the option, however, is out of the money there is a large reduction in the loss criterion. In the simulation study out

| $T$ | $K$ | $k$ | BS | GARCH | TGARCH |
|---|---|---|---|---|---|
| | 1550 | 10 | 0.017 | 0.014 | 0.014 |
| Jan | 1600 | 10 | 0.099 | 0.029 | 0.028 |
| | 1650 | 10 | 4.231 | 1.626 | 1.314 |
| | 1600 | 47 | 1.112 | 0.961 | 0.954 |
| Mar | 1650 | 53 | 1.347 | 1.283 | 1.173 |
| | 1700 | 56 | 1.827 | 1.701 | 1.649 |
| | 1600 | 53 | 1.385 | 1.381 | 1.373 |
| Jun | 1650 | 56 | 2.023 | 1.678 | 1.562 |
| | 1700 | 51 | 2.460 | 2.053 | 1.913 |
| Sum | | 346 | 14.500 | 10.725 | 9.980 |

Table 15.4: The loss criterium $U$ for the DAX calls with maturity at $T$ and a strike price $K$ using BS, GARCH and TGARCH option prices. The number of observations is given by $k$.

of the money options react the most sensitive to stochastic volatility and the leverage effect. In the situation with real data this is most obvious for the January-1650 option, where Black-Scholes performs poorly and TGARCH performs better than GARCH. For the March and June options the difference is not so obvious. This can be explained by the fact that the index increased to a level of 1736 points on 20 March of 1717 points on 30 March, so that the option with a strike price of 1700 became in the money. This is also the explanation for the fact that $U$ is the highest for the January-1650 option. There were only 10 trading days, but the option was out of the money for several days. For example, the DAX closed on 8 January at 1578 points.

Since in every case TGARCH performs better than GARCH, we conclude that the market follows the asymmetry of the volatility. Therefore, specifying the volatility model correctly plays an important role in determining option prices.

## 15.4 Recommended Literature

The presentation of this chapter closely follows the work of Härdle and Hafner (2000). The standard ARCH model originated in Engle (1982), the development of EGARCH in Nelson (1991) and TGARCH in Zakoian (1994) (for the standard deviation) and Glosten et al. (1993) (for the variance). Nonparametric and semi-parametric variants of the ARCH model were suggested

and studied by Carroll, Härdle and Mammen (2002) and Hafner (1998). The classic option pricing model with stochastic volatility originated in Hull and White (1987). Hull and White implicitly assume that the market price of the risk of the volatility is zero, whereas in Melino and Turnbull (1990) it is different from zero, constant and exogenous. Empirical evidence for the valuation of risk of the volatility is given in Wiggins (1987). Renault and Touzi (1996) generalise the model from Hull and White (1987), in that they allow a market price of the risk for the volatility, which itself can vary over time. The concept of minimising the quadratic loss of a hedge portfolio is given in Föllmer and Sondermann (1991) and Föllmer and Schweizer (1991). The practical procedure to implement "15 minute old" implied volatility into the Black/Scholes formula, was successfully used in Bossaerts and Hillion (1993).

# 16 Value at Risk and Backtesting

The Value-at-Risk (VaR) is probably the most known measure for quantifying and controlling the risk of a portfolio. The establishment of the VaR is of central importance to a credit institute, since it is the basis for a regulatory notification technique and for required equity investments. The description of risk is done with the help of an "internal model", whose job is to reflect the market risk of portfolios and similar risky investments over time. This often occurs though the choice of suitable portfolios of a specific risk factor, i.e., through principal components analysis (Chapter 20). With risks from option trading a linear transformation is often applied using the "Greeks" (Chapter 6).

The objective parameter in the model is the probability forecasts of portfolio changes over a given time horizon. Whether the model and its technical application correctly identify the essential aspects of the risk, remains to be checked. The backtesting procedure serves to evaluate the quality of the forecast of a risk model in that it compares the actual results to those generated with the VaR model. For this the daily VaR estimates are compared to the results from hypothetical trading that are held from the end-of-day position to the end of the next day, the so called "clean backtesting". The concept of clean backtesting is differentiated from that of "mark-to-market" profit and loss ("dirty $P\&L$") analyses in which intra-day changes are also observed. In judging the quality of the forecast of a risk model it is advisable to concentrate on the clean backtesting.

The interest of an institute in a correct VaR calculation can be traced back to a rule of equity investing, which we will briefly describe here. Since 1997 (modification of the Basle market risk paper) institutes have been allowed to replicate specific risks using internal models. Included here under specific risks are those associated with daily price developments ("residual risks") and others realised from rare occurrences ("event risks" such as rating changes). Models that only consider residual risks are called "surcharge model", and those that consider event risks are called "non-surcharge model". For calculating capital investments the institutes have to use the following formula,

| Exceedances | Increase of M | Zone |
|---|---|---|
| 0 bis 4 | 0 | **green** |
| 5 | 0.4 | yellow |
| 6 | 0.5 | **yellow** |
| 7 | 0.65 | **yellow** |
| 8 | 0.75 | **yellow** |
| 9 | 0.85 | **yellow** |
| More than 9 | 1 | **red** |

Table 16.1: Traffic light as a factor of the exceeding amount.

Graumert and Stahl (2001):

$$EMU_t = max\{VaR_{t-1} + d \cdot SR_{t-1}; M \cdot \frac{1}{60}\sum_{i=1}^{60} VaR_{t-i} + d \cdot \frac{1}{60}\sum_{i=1}^{60} SR_{t-i}\} \tag{16.1}$$

- $EMU_t$ = Capital investment for the price risks determined by the risk model at day $t$
- $VaR_{t-i}$ = VaR estimation at day $t-i$ for the general and the specific price risk
- $d$ = Indicator variable with $d = 1$ for surcharge models and $d = 0$ for non-surcharge models and for models that only model general risk
- $M$ = Multiplier with $M = 3 + ZBT + ZQM$
- $ZBT$ = Backtesting surcharge factor according to § 37 $II$ GI ($0 \leq ZBT \leq 1$)
- $ZQM$ = Surcharge factor for qualitative deficiencies ($0 \leq ZQM \leq 1$)
- $SR_{t-i}$ = Surcharge for not modelling event risk in surcharge models on day $t-i$

The multiplier $M$ in (16.1) contains the backtesting surcharge factor which is calculated from the so called "traffic light". According to the "traffic light" the value $M$ increases with the number of years the VaR values exceeds the actual loss. Table 16.1 explains the "traffic light" zones.

## 16.1 Forecast and VaR Models

Value at Risk (VaR) models are used in many financial applications. Their goal is to quantify the profit or loss of a portfolio which could occur in the near future. The uncertainty of the development of a portfolio is expressed in a "forecast distribution" $P_{t+1}$ for period $t+1$.

$$P_{t+1} = \mathcal{L}(L_{t+1}|\mathcal{F}_t)$$

is the conditional distribution of the random variable $L_{t+1}$, which represents the possible profits and losses of a portfolio in the following periods up to date $t+1$, and $\mathcal{F}_t$ stands for the information in the available historical data up to date $t$. An estimator for this distribution is given by the forecast model. Consequently the possible conditional distributions of $L_{t+1}$ come from a parameter class $\mathcal{P}_{t+1} = \{P_{t+1}^{\theta(t)} \,|\, \theta(t) \in \Theta\}$. The finite-dimensional parameter $\theta(t)$ is typically estimated from $n = 250$ historical return observations at time $t$, that is approximately the trading days in a year. Letting $\hat{\theta}(t)$ stand for this estimator then $\mathcal{L}(L_{t+1}|\mathcal{F}_t)$ can be approximated with $P_{t+1}^{\hat{\theta}(t)}$.

An important example of $\mathcal{P}_{t+1}$ is the Delta-Normal Model, RiskMetrics (1996). In this model we assume that the portfolio is made up of $d$ linear (or linearised) instruments with market values $X_{k,t}, k = 1, ..., d$, and that the combined conditional distribution of the log returns of the underlying

$$Y_{t+1} \in \mathbb{R}^d, Y_{k,t+1} = \ln X_{k,t+1} - \ln X_{k,t}, k = 1, ..., d,$$

given the information up to time $t$ is a multivariate normal distribution, i.e.,

$$\mathcal{L}(Y_{t+1}|\mathcal{F}_t) = \mathrm{N}_d(0, \Sigma_t) \tag{16.2}$$

where $\Sigma_t$ is the (conditional) covariance matrix of the random vector $Y_{t+1}$. We consider first a single position ($d = 1$), which is made up of $\lambda_t$ shares of a single security with an actual market price $X_t = x$. With $w_t = \lambda_t x$ we represent the *exposure* of this position at time $t$, that is its value given $X_t = x$. The conditional distribution of the changes to the security's value $L_{t+1} = \lambda_t(X_{t+1} - X_t)$ is approximately:

$$\begin{aligned} \mathcal{L}(L_{t+1}|\mathcal{F}_t) &= \mathcal{L}(\lambda_t(X_{t+1} - x) \mid \mathcal{F}_t) \\ &= \mathcal{L}(w_t \frac{X_{t+1} - x}{x} \mid \mathcal{F}_t) \\ &\approx \mathcal{L}(w_t Y_{t+1} \mid \mathcal{F}_t) = \mathrm{N}(0, w_t^2 \sigma_t^2) \end{aligned} \tag{16.3}$$

with $\sigma_t^2 = \mathrm{Var}(Y_{t+1} \mid \mathcal{F}_t)$. Here we have used the Taylor approximation

$$\ln X_{t+1} - \ln x = \frac{X_{t+1} - x}{x} + o(X_{t+1} - x). \tag{16.4}$$

The generalisation to a portfolio that is made up of $\lambda_t^1, \cdots, \lambda_t^d$ shares of $d$ (linear) instruments is quite obvious. Let $w_t$ be the $d$-dimensional exposure vector at time $t$

$$w_t = (w_t^1, \cdots, w_t^d)^\top = (\lambda_t^1 x^1, \cdots, \lambda_t^d x^d)^\top. \tag{16.5}$$

$$L_{t+1} = \sum_{k=1}^{d} \lambda_t^k (X_{k,t+1} - X_{k,t})$$

is the change in the value of the portfolio. For a single position the conditional distribution of $L_{t+1}$ given the information $\mathcal{F}_t$ is approximately equal to the conditional distribution of

$$w_t^\top Y_{t+1} = \sum_{k=1}^{d} w_t^k Y_{k,t+1}.$$

In the framework of Delta-Normal models this distribution belongs to the family

$$\mathcal{P}_{t+1} = \{\mathrm{N}(0, \sigma_t^2) : \sigma_t^2 \in [0, \infty)\}, \tag{16.6}$$

with $\sigma_t^2 = w_t^\top \Sigma_t w_t$. The goal of the VaR analysis is to approximate the parameter $\theta(t) = \sigma_t$ and thus to approximate the forecast distribution of $P_{t+1}$.

Now consider the problem of estimating the forecast distribution from the view point of the following model's assumptions. The change in the value of the portfolio is assumed to be of the form

$$L_{t+1} = \sigma_t \, Z_{t+1} \tag{16.7}$$

$$\sigma_t^2 = w_t^\top \Sigma_t w_t, \tag{16.8}$$

where $Z_t$ is i.i.d. N(0,1) distributed random variable, $w_t$ is the exposure vector at time $t$ and $\Sigma_t$ is the (conditional) covariance matrix of the vector $Y_{t+1}$ of the log returns. We combine the last $n$ realisations of $Y_t = y_t, \ldots, Y_{t-n+1} = y_{t-n+1}$ from the log return vector with a $(n \times d)$ matrix $\mathcal{Y}_t = (y_i^\top)_{i=t-n+1,\ldots,t}$. From these observations we calculate two estimators from $\Sigma_t$; first the naive RMA, i.e., *rectangular moving average*:

$$\hat{\Sigma}_t = \frac{1}{n} \mathcal{Y}_t^\top \mathcal{Y}_t. \tag{16.9}$$

Since the expected value of the vector of returns $Y_t$ is zero according to the Delta-Normal model, this is exactly the empirical covariance matrix. The second so called EMA estimator, i.e., *exponentially moving average*, is based

on an idea from Taylor (1986) and uses an exponential weighting scheme. Define for $\gamma,\ 0 < \gamma < 1$

$$\tilde{y}_{t-k} = \gamma^k y_{t-k}, k = 0, ..., n-1, \quad \tilde{\mathcal{Y}}_t = (\tilde{y}_i^\top)_{i=t-n+1,...,t}$$

a log return vector is exponentially weighted over time and a $(n \times d)$ matrix is constructed from this, then $\Sigma_t$ is estimated with

$$\hat{\Sigma}_t = (1-\gamma)^{-1} \tilde{\mathcal{Y}}_t^\top \tilde{\mathcal{Y}}_t. \tag{16.10}$$

This normalisation makes sense, since the sum $\sum_{i=1}^n \gamma^{i-1} = \frac{1-\gamma^n}{1-\gamma}$ for $\gamma \to 1$ converges to $n$, thus the RMA estimator is the boundary case of the EMA estimator. Both estimators can be substituted in (16.7) and (16.8), and we obtain with

$$\hat{P}_{t+1} = \mathrm{N}(0, \hat{\sigma}_t^2), \quad \hat{\sigma}_t^2 = w_t^\top \hat{\Sigma}_t w_t$$

an approximation of the forecast distribution, i.e., the conditional distribution of $L_{t+1}$. It should be noted that the Bundesanstalt für Finanzdienstleistungsaufsicht (http://www.bafin.de) currently dictates the RMA technique.

The *Value at Risk* VaR is determined for a given level $\alpha$ by

$$VaR_t = F_{t+1}^{-1}(\alpha) \stackrel{\text{def}}{=} inf\{x; F_{t+1}(x) \geq \alpha\} \tag{16.11}$$

and estimated with

$$\widehat{VaR}_t = \hat{F}_{t+1}^{-1}(\alpha) \stackrel{\text{def}}{=} inf\{x; \hat{F}_{t+1}(x) \geq \alpha\}. \tag{16.12}$$

Here $F_{t+1}, \hat{F}_{t+1}$ represent the distribution function of $P_{t+1}$, $\hat{P}_{t+1}$. The quality of the forecast is of particular interest in judging the VaR technique. It can be empirically checked using the realised values $(\hat{P}_t, L_t), t = 1, ..., N,$. In the event that the model assumptions, for example, (16.7) and (16.8), are correct for the form of the forecast's distribution, then the sample $U_t = F_t(L_t), t = 1, ..., N$, should have independent uniformly distributed random values over the interval $[0, 1]$ and $\hat{U}_t = \hat{F}_t(L_t), t = 1, ..., N$, approximately independent identically uniformly distributed random values. Then the ability of the forecasts distribution to fit the data is satisfied.

## 16.2 Backtesting with Expected Shortfall

Below we consider the expected shortfall from $L_{t+1}$ as an alternative to the VaR and develop a backtesting method for this risk measurement. The *expected shortfall*, also called the *Tail-VaR*, is in the Delta-Normal Model, i.e.

under the assumptions from (16.7) and (16.8), defined by

$$\begin{aligned} E(L_{t+1} \mid L_{t+1} > VaR_t) &= E(L_{t+1} \mid L_{t+1} > z_\alpha \ \sigma_t) \\ &= \sigma_t \ E(L_{t+1}/\sigma_t \mid L_{t+1}/\sigma_t > z_\alpha). \end{aligned} \tag{16.13}$$

Here $z_\alpha = \Phi^{-1}(\alpha)$ represents the $\alpha$ quantile of the standard normal distribution, where $\Phi$ is the standard normal distribution function.

Under this model (16.7) and (16.8) $Z_{t+1} = L_{t+1}/\sigma_t$ has a standard normal distribution. For a defined threshold value $u$ we obtain

$$\vartheta = E(Z_{t+1} \mid Z_{t+1} > u) = \frac{\varphi(u)}{1 - \Phi(u)} \tag{16.14}$$

$$\varsigma^2 = Var(Z_{t+1} \mid Z_{t+1} > u) = 1 + u \cdot \vartheta - \vartheta^2, \tag{16.15}$$

where $\varphi$ is the standard normal density. For given observations from a forecast distribution and its realisations $(\hat{F}_{t+1}(\cdot/\hat{\sigma}_t), L_{t+1}/\hat{\sigma}_t)$ we consider (16.14) as the parameter of interest. Replacing the expected value with a sample mean and the unobservable $Z_{t+1}$ with

$$\hat{Z}_{t+1} = \frac{L_{t+1}}{\hat{\sigma}_t}, \tag{16.16}$$

where $\sigma_t$ in (16.8) is estimated with (16.9) or (16.10), we obtain an estimator for $\vartheta$

$$\hat{\vartheta} = \frac{1}{N(u)} \sum_{t=0}^{n} \hat{Z}_{t+1} \ \mathbf{1}(\hat{Z}_{t+1} > u). \tag{16.17}$$

$N(u)$ is the random number of times that the threshold value $u$ is exceeded:

$$N(u) = \sum_{t=1}^{n} \mathbf{1}(\hat{Z}_{t+1} > u).$$

Inferencing on the expected shortfall, i.e., on the difference $\hat{\vartheta} - \vartheta$, we obtain the following asymptotical result:

$$\sqrt{N(u)}\Big(\frac{\hat{\vartheta} - \vartheta}{\hat{\varsigma}}\Big) \xrightarrow{\mathcal{L}} \mathrm{N}(0,1) \tag{16.18}$$

(16.18) can be used to check the adequacy of the Delta-Normal model.

## 16.3 Backtesting in Action

The data used in this section is a bond portfolio of a German bank from 1994 to 1995. The portfolio is not adjusted so that the exposure vector $w_t = w$ is

time dependent. We assume that (16.7) and (16.8) hold. The VaR forecast is based on both prediction rules introduced in Section 16.1 that are used to estimate the parameters $\sigma_t$ of the forecast distribution in RMA and EMA given $\gamma = 0.94$. In light of the bond crisis in 1994 it is interesting how both techniques respond to this stress factor.

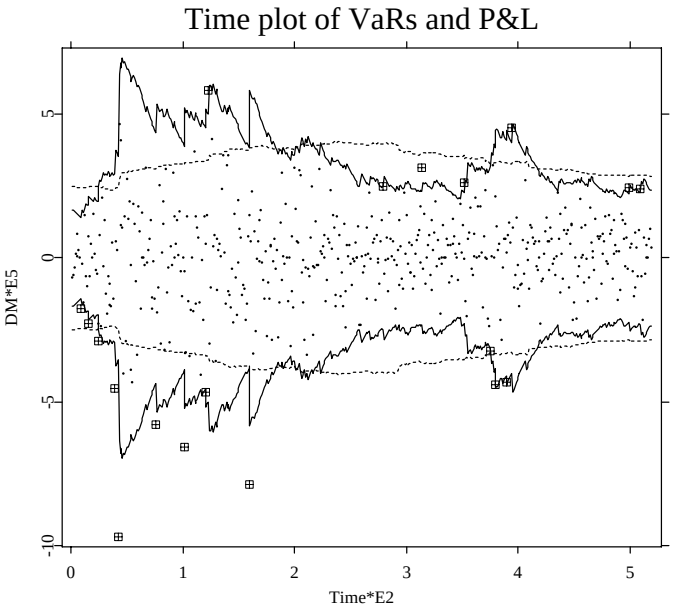


Figure 16.1: The dots show the observed changes $L_t$ in the value of the portfolio. The dashed line represents the forecasted VaRs based on RMA (99% and 1%). The solid line represents the same for EMA. SFEVaRbank SFEVaRtimeplot

The significance level under consideration is $\alpha = 1\%$ for large losses and $\alpha = 99\%$ for large profits. To investigate we include plots of time series from the realised P/L (i.e., profit-loss) data $L_t$ as compared to the respective VaR estimator $\widehat{VaR_t}$ calculated with (16.12). If the model and the estimation of the parameter $\sigma_t$ based on forecast distribution are adequate, then approximately 1% of the data should lie below the 1% and above the 99% VaR Estimators. In addition in Figure 16.1 the crossings for the case where VaR is estimated with EMA are marked. We recognise that in 1994 (1995) there were a total of 10 (9) crossings determined for the EMA method. This strongly contrasts the 17 (3) observed values for the RMA Method. It is clear

that the RMA technique leads to, above all during the bond crisis in 1994, too many crossings for the 1% VaR estimator, which means that the probability of larger losses is underestimated. This tendency to underestimate the risk is produced from the observation width of 250 days, when the market is moving towards a more volatile phase. The opposite is true when moving in the other direction; RMA overestimates risk. The EMA adapts more quickly to market phases since data in the past has less influence on the estimator due to the exponentially deteriorating weights. With SFEVaRtimeplot we have calculated the estimated VaRs for another bank using the EMA and RMA respectively.

The poor forecast quality of the RMA, in particular for the left side of the distribution, can also be seen in that for a particular day the VaR was exceeded by 400%. If the model (16.7) - (16.8) is correct, then the variable (16.19) must have a standard deviation of about 0.41. The empirical standard deviation calculated from the data is about 0.62. According to the volatility scale of the RMA the risk is underestimated on average by $\frac{0.62-0.41}{0.41} \approx 50\%$. The EMA plot in Figure 16.1 shows a better calibration. The empirical standard deviation of (16.19) is in this case around 0.5, which corresponds to an underestimation of risk by approximately 25%.

All other diagnostic measurements are entered into the QQ plot of the variable

$$\frac{L_{t+1}}{\widehat{VaR}_t} = \frac{L_{t+1}}{2.33\hat{\sigma}_t}, \tag{16.19}$$

see Figure 16.2 and Figure 16.3. If the VaR forecast $\widehat{VaR}_t$ was perfect, the QQ plot would produce a straight line and fill out the area in $[-1, 1]$.

A comparison of the graphs in Figure 16.2 and Figure 16.3 show that the EMA method is calibrated better than the RMA method. The RMA method clearly shows outliers at both ends. The interval boundaries of $[-1, 1]$ are in both cases clearly exceeded. This indicates a possible inadequacy of an assumed normal distribution. QQ plots for the year 1995 are not shown, which also clearly show the dominance of EMA over RMA.

Another important assumption of our model is the independence of the rescaled random variable $Z_t$. Figure 16.4 shows the outliers of another bank

$$\{t, \mathbf{1}(L_{t+1} > \widehat{VaR}_t)\}, \ t = 1, ..., 750, \tag{16.20}$$

as a function of $t$. The contradictory temporal non-uniform distribution of the outliers from the independence of $Z_t$ is clearer to see by the RMA method than by the EMA method.

The exploratory analysis clearly shows the differences between RMA and

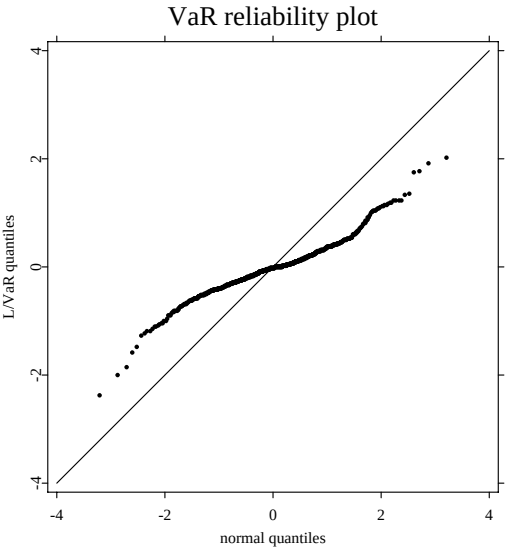


Figure 16.2: QQ plot of $L_{t+1}/\widehat{VaR}_t$ for RMA in 1994. SFEVaRqqplot

EMA. As a supplement we now compare both estimation techniques with an appropriate test within the framework of the model (16.7) - (16.8). We again consider the sample residuals $\hat{Z}_{t+1}$ from (16.16) and set the threshold value in (16.14) to $u = 0.8416$, i.e., to the 80% quantile of the distribution of $Z_{t+1} = \frac{L_{t+1}}{\sigma_t}$. From this we obtain $\vartheta = 1.4$ according to (16.14). Due to the asymptotic distribution (16.18) we can check the significance of the hypothesis

$$H_0 \ : \vartheta \overset{(<)}{=} 1.4. \tag{16.21}$$

A better approximation than the standard normal distribution for the sample is the Student $t(20)$ distribution, if we generalise the degrees of freedom.

$$\mathcal{L}(\hat{Z}_{t+1}) = \mathcal{L}\big(\frac{L_{t+1}}{\hat{\sigma}_t}\big) \approx t(20). \tag{16.22}$$

The value of $\vartheta$ obtained differs from the value given above by 5%, the corresponding variances $\varsigma^2$ by 18%. Therefore, we also consider the hypothesis

$$H_0 \ : \vartheta \overset{(<)}{=} 1.47. \tag{16.23}$$

The following Table 16.2 to Table 16.5 summarises our results.

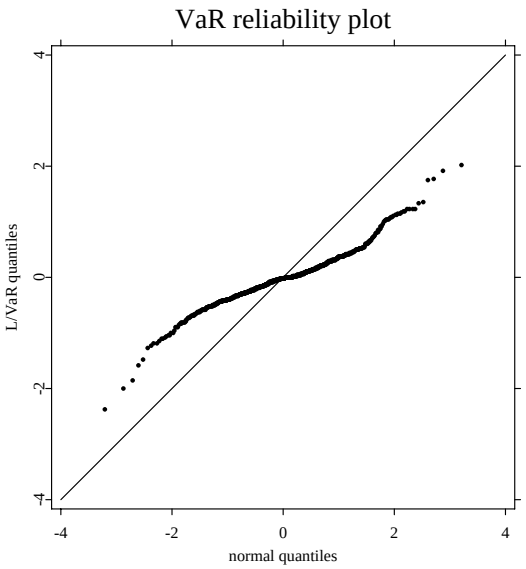


Figure 16.3: QQ plot of $L_{t+1}/\widehat{VaR}_t$ for EMA in 1994. SFEVaRqqplot

From Table 16.2 and Table 16.3 it is obvious that the observed outliers for EMA are calibrated better than for the RMA method. For a random sample of 260 values we expect 52 outliers (standard deviation 6.45). For EMA we observe 61 $(61 - 52 \approx 1.5 \cdot$ standard deviation) outliers and for RMA 68 $(68 - 52 \approx 2.5 \cdot$ standard deviation). Naturally the outliers influence the test considerably. We can therefore repeat the analysis excluding the outliers and obtain (16.4) and (16.5).

To conclude we can say that the EMA method gives better calibrated results

| Method | $\vartheta = 1.4$ | $\varsigma = 0.46$ | $\sqrt{N(u)}\frac{\hat{\vartheta}-\vartheta}{\hat{\varsigma}}$ | significance | $N(u)$ |
|---|---|---|---|---|---|
| EMA | $\hat{\vartheta} = 1.72$ | $\hat{\varsigma} = 1.01$ | 2.44 | 0.75% | 61 |
| RMA | $\hat{\vartheta} = 1.94$ | $\hat{\varsigma} = 1.3$ | 3.42 | 0.03% | 68 |

Table 16.2: $H_0 : \vartheta \overset{(<)}{=} 1.4$

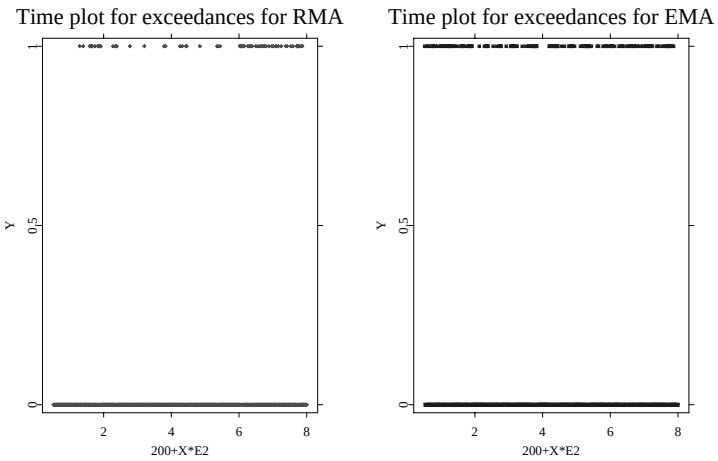


Figure 16.4: Time diagram of the exceedances at the 80% significance level from VaR for RMA (left) and EMA. The superiority of EMA is obvious. SFEVaRtimeplot2

| Method | $\vartheta = 1.47$ | $\varsigma = 0.546$ | $\sqrt{N(u)}\frac{\hat{\vartheta}-\vartheta}{\hat{\varsigma}}$ | significance | $N(u)$ |
|---|---|---|---|---|---|
| EMA | $\hat{\vartheta} = 1.72$ | $\hat{\varsigma} = 1.01$ | 2.01 | 2.3% | 61 |
| RMA | $\hat{\vartheta} = 1.94$ | $\hat{\varsigma} = 1.3$ | 3.04 | 0.14% | 68 |

Table 16.3: $H_0 : \vartheta \overset{(<)}{=} 1.47$

than the RMA method. Both methods are extremely sensitive to outliers and should both be considered. Even the EMA method suffers from the assumptions (16.7) - (16.8), which are based on the Delta-Normal Model, can only be approximately fulfilled. The residuals $Z_t$ are neither normally distributed nor independent, although the EMA method is not strongly effected by the independence assumption due to its exponentially decreasing memory.

## 16.4 Recommended Literature

The classical start to Value at Risk (VaR) estimation lies in the consideration of linear or linearised portfolios, see RiskMetrics (1996). The linear structure

| Method | $\vartheta = 1.4$ | $\varsigma = 0.46$ | $\sqrt{N(u)}\frac{\hat{\vartheta}-\vartheta}{\hat{\varsigma}}$ | significance | $N(u)$ |
|---|---|---|---|---|---|
| EMA | $\hat{\vartheta} = 1.645$ | $\hat{\varsigma} = 0.82$ | 2.31 | 1% | 60 |
| RMA | $\hat{\vartheta} = 1.83$ | $\hat{\varsigma} = 0.93$ | 3.78 | 0.00% | 67 |

Table 16.4: $H_0 : \vartheta \overset{(<)}{=} 1.4$ largest outlier excluded

| Method | $\vartheta = 1.47$ | $\varsigma = 0.546$ | $\sqrt{N(u)}\frac{\hat{\vartheta}-\vartheta}{\hat{\varsigma}}$ | significance | $N(u)$ |
|---|---|---|---|---|---|
| EMA | $\hat{\vartheta} = 1.645$ | $\hat{\varsigma} = 0.82$ | 1.65 | 5% | 60 |
| RMA | $\hat{\vartheta} = 1.83$ | $\hat{\varsigma} = 0.93$ | 3.1 | 0.15% | 67 |

Table 16.5: $H_0 : \vartheta \overset{(<)}{=} 1.47$ largest outlier excluded

transforms the multi-dimensional normally distributed random variables into one dimensional Gaussian values whose quantile can be estimated. An introduction to the asymptotic distributions of extreme values can be found in Leadbetter, Lindgren and Rootzen (1983) and Embrechts et al. (1997). McAllister and Mingo (1996) describe the advantages from (16.13) in a RAROC (risk-adjusted return on capital) setup. Artzner and Heath (1997) claim that the expected shortfall is a coherent measurement of risk. Jaschke and Küchler (1999) show that (16.13) is a reasonable approximation for a worst case scenario. Leadbetter et al. (1983) show how (16.13) can be used in the context of the theory of extreme values. A good overview of the VaR problems is given in Jorion (2000). The majority of the German laws can be found under http://www.bafin.de. Taleb (2001) is a critic of specific VaR definitions and gives several examples in which Value at Risk definitions can be "blinding" given certain trading strategies ("Peso Problem Traders"). A complete literature review can be found in Franke, Härdle and Stahl (2000). The VaR calculations from Overbeck (2000) based on the ability to pay process are discussed and country risk is evaluated in Lehrbass (2000).

# 17 Copulae and Value at Risk

The capital requirement from financial institutions is based on the amount of risk carried in their portfolios. The risk associated with a portfolio may originate from:

1. fluctuations in the value of financial assets composing the portfolio (*market* risk),
2. fluctuations in the credibility of debtors (*credit* risk),
3. uncertainty connected with technical, personal and natural factors that may influence the portfolio value (*operational* risk).

In order to investigate the risk of a portfolio, the assets subjected to risk (*risk factors*) should be identified and the changes in the portfolio value caused by the risk factors evaluated. Especially relevant for risk management purposes are *negative changes* - the portfolio *losses*.

The *Value-at-Risk* (VaR) is a measure that quantifies the riskiness of a portfolio. This measure and its accuracy are of crucial importance in determining the capital requirement from financial institutions. That is one of the reasons why increasing attention has been paid to VaR computing methods.

The losses and the probabilities associated with them (the *distribution of losses*) are necessary to describe the degree of portfolio riskness. The riskier the portfolio, the higher the probability of losses being larger than a certain amount is. In other words, the riskier the portfolio, the larger are the minimal losses for a certain probability (also called *level*). That is the precise of VaR definition: VaR is a quantile of the distribution of portfolio losses representing the minimal losses for a certain level.

Looking carefully at the distribution of losses, one verifies that large losses are influenced by simultaneous losses in risk factors. Hence, the *distribution of losses depends on joint distribution of risk factors*.

Understanding the joint distribution of risk factors is fundamental in investigating and computing the Value-at-Risk. The conventional procedure to model joint distributions of financial returns is to approximate them with *multivariate normal distributions*.

That implies, however, that the dependence structure of the returns is reduced to a fixed type. Even if the autocorrelation structure is neglected, predetermining a multivariate normal distribution means that the following assumptions hold:

1. symmetric distribution of returns,
2. the tails of the distribution are not too heavy,
3. linear dependence.

Empirical evidence for these assumptions are barely verified and an alternative model is needed, with more flexible dependence structure and arbitrary marginal distributions. These are exactly the characteristics of *copulae*.

Copulae are very useful for modelling and estimating multivariate distributions. The flexibilty of copulae basically follows on from *Sklar's Theorem*, which says that each joint distribution can be"decomposed" into its marginal distributions and a copula $C$ "responsible" for the dependence structure:

$$F(x_1 \ldots, x_d) \quad = \quad C\{F_1(x_1), \ldots, F_d(x_d)\}.$$

Two important facts for practical applications rely on this theorem:

1. the construction of multivariate distributions may be done in two independent steps: the specification of marginal distributions - not necessarily identical - and the specification of a dependence structure. Copulae "couple together" the marginal distributions into a multivariate distribution with the desired dependence structure.
2. joint distributions can be separately estimated from a sample of observations: the marginal distributions are estimated first, the dependence structure later.

The copula approach gives us more freedom than the normality assumptions: marginal distributions with asymmetric heavy tails (typical for financial returns) can be combined with different dependence structures, resulting in multivariate distributions (far different from the multivariate normal) that better describe the empirical characteristics of financial returns distribution.

Moreover, copulae allow for dynamical modelling and adaption to portfolios: different copulae with distinct properties can be associated to different portfolios according to their specific dependence structures. Furthermore, copulae may change as time evolves, reflecting the evolution of the dependence between financial assets. To summarise, the Value-at-Risk estimation with copulae is more efficient and flexible than any method based on normality assumption.

## 17.1 Copulae

This section presents the basic copulae definitions and theorems. The most important copulae, together with their standard construction and simulation methods are also discussed.

**Definition 17.1 (Copula)**
*A $d$-dimensional copula is a function $C : [0,1]^d \to [0,1]$ satisfying the following properties for every $u = (u_1, \ldots, u_d)^\top \in [0,1]^d$ and $j \in \{1, \ldots, d\}$:*

1. *if $u_j = 0$ then $C(u_1, \ldots, u_d) = 0$*
2. *$C(1, \ldots, 1, u_j, 1, \ldots, 1) = u_j$*
3. *for every $v = (v_1, \ldots, v_d)^\top \in [0,1]^d$, $v_j \leq u_j$*

$$V_C(u, v) \geq 0$$

*where $V_C(u, v)$ is given by*

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_i + \ldots + i_d} C(g_{1i_1}, \ldots, g_{di_d})$$

*and $g_{j1} = v_j$ and $g_{j2} = u_j$.*

The first and third properties state that copulae are grounded functions and that all $d$-dimensional boxes with vertices in $[0,1]^d$ have *non-negative $C$-volume*. Together they guarantee that copulae are distribution functions on the $d-$dimensional unit cube, while the second property reveals that copulae have *uniform marginal distributions*.

Note that by considering random variables $X_1, \ldots, X_d$ with univariate distribution functions $F_{X_1}, \ldots, F_{X_d}$ and the random variables $U_i = F_{X_i}(X_i)$, $i = 1, \ldots, d$ uniform distributed in $[0,1]$, a copula may be interpreted as *the joint distribution of the marginal distributions*.

For all $u = (u_1, \ldots, u_d)^\top \in [0,1]^d$, every copula $C$ satisfies

$$W(u_1, \ldots, u_d) \leq C(u_1, \ldots, u_d) \leq M(u_1, \ldots, u_d)$$

where

$$M(u_1, \ldots, u_d) = \min(u_1, \ldots, u_d) \tag{17.1}$$

and

$$W(u_1, \ldots, u_d) = \max\left(\sum_{i=1}^{d} u_i - d + 1, 0\right). \tag{17.2}$$

$M(u_1, \ldots, u_d)$ is called *Fréchet-Hoeffding upper bound* and $W(u_1, \ldots, u_d)$ the *Fréchet-Hoeffding lower bound*. While $M$ is not a copula for $d > 2$, $W$ is a copula for all $d$. Besides the Fréchet-Hoeffding bounds, the product copula $\Pi(u_1, \ldots, u_d)$ is of fundamental importance. The product copula is given by:

$$\Pi(u_1, \ldots, u_d) = \prod_{j=1}^{d} u_j. \tag{17.3}$$

Figure 17.1 illustrates the Fréchet-Hoeffding bounds and the product copulae.

The following theorem connects copulae with distribution functions and shows that:

- every distribution function can be "decomposed" into its marginal distribution and (at least) one copula.
- a (unique) copula is obtained from "decoupling" every (continuous) multivariate distribution function from its marginal distributions.

**Theorem 17.1 (Sklar's theorem)**
*Let $F$ be a $d$-dimensional distribution function with marginals $F_1 \ldots, F_d$. Then a copula $C$ with*

$$F(x_1, \ldots, x_d) = C\{F_1(x_1), \ldots, F_d(x_d)\} \tag{17.4}$$

*can exist for every $x_1, \ldots, x_d \in \overline{\mathbb{R}}$. If $F_1, \ldots, F_d$ are continuous, then $C$ is unique. On the other hand, if $C$ is a copula and $F_1, \ldots, F_d$ are distribution functions, then the function $F$ defined in (17.4) is a joint distribution function with marginals $F_1, \ldots, F_d$.*

Hence, for a joint distribution $F$ with continuous marginals $F_1, \ldots, F_d$ the unique copula $C$ can be obtained from (17.4) for all $u = (u_1, \ldots, u_d)^\top \in [0, 1]^d$ as

$$C(u_1, \ldots, u_d) = F\{F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\}. \tag{17.5}$$

**Definition 17.2 (Copula of a random variable)**
*Let $X = (X_1, \ldots, X_d)^\top$ be a random vector with distribution $X \sim F_X$ and continuous marginals $X_j \sim F_{X_j}$. The copula of $X$ is the distribution function $C_X$ of $u = (u_1, \ldots, u_d)^\top$ where $u_j = F_{X_j}(x_j)$:*

$$C_X(u_1, \ldots, u_d) = F_X\{F_{X_1}^{-1}(u_1), \ldots, F_{X_d}^{-1}(u_d)\}. \tag{17.6}$$

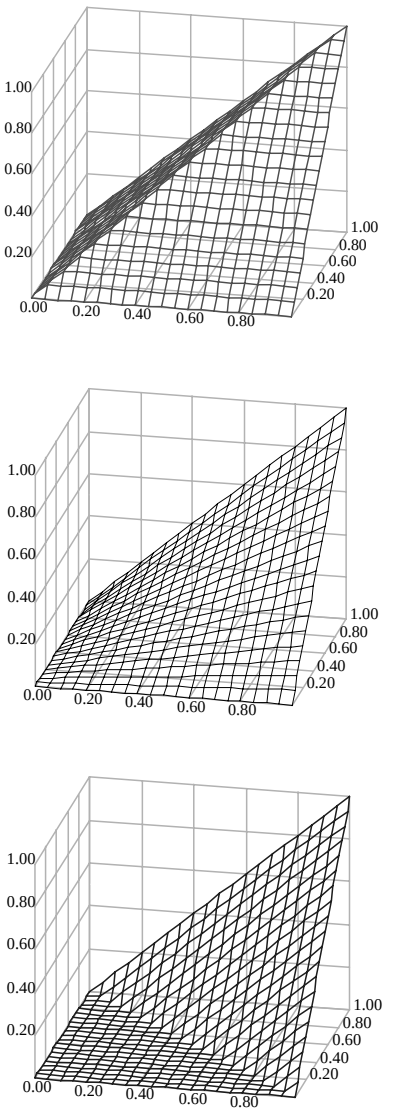


Figure 17.1: Fréchet-Hoeffding upper bound $W(u_1, u_2)$ (upper panel), product copula $\Pi(u_1, u_2)$ (middle panel), Fréchet-Hoeffding lower bound $M(u_1, u_2)$ (lower panel).

SFEfrechet

For independent random variables $X_1, \ldots, X_d$ the copula of $X$ is the product copula defined in (17.3):

$$\begin{aligned} C_X(u_1, \ldots, u_d) &= F_X(x_1, \ldots, x_d) \\ &= \prod_{j=1}^{d} F_{X_j}(x_j) \\ &= \Pi\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\} \\ &= \Pi(u_1, \ldots, u_d). \end{aligned}$$

Note that the product copula is the same for *any* marginal distribution, i.e., it determines *the dependence structure* between the univariate variables for arbitrary marginals. The next theorem shows that copulae are *invariant under monotone increasing transformations*. This property is very useful for obtaining copula families in subsequent sections.

**Theorem 17.2**
*Let $X = (X_1, \ldots, X_d)^\top$ be a random vector with continuous marginals and copula $C_X$ and $T_1, \ldots, T_d$ be strictly increasing functions on range$(X_1)$,..., range$(X_d)$. Let $Y = (Y_1, \ldots, Y_d)^\top$, $Y_i = T_i(X_i)$ be a random vector with copula $C_Y$. Then $C_X = C_Y$ almost everywhere.*

A $d$-dimensional random variable determines a copula through its joint and marginal distributions. Moreover, monotone increasing transformations on the random variable do not affect the copula. These are the main ideas used to obtain the Gaussian copula: the random variable $X = (X_1, \ldots, X_d)^\top$ with multivariate normal distribution and copula $C_X$ is transformed into the standardised variable $Z = (Z_1, \ldots, Z_d)^\top$, $Z_j \sim N(0, 1)$. The copula of the random variable $Z$ is $C_X$.

A *copula density* exists for an *absolute continuous* copula there. Copula densities are essential for estimation procedures, as seen in Section 17.2.

**Definition 17.3 (Copula density)**
*For an absolutely continuous copula $C$, the copula density is defined as*

$$c(u_1, \ldots, u_d) = \frac{\partial^d C(u_1, \ldots, u_d)}{\partial u_1 \ldots \partial u_d}. \tag{17.7}$$

Given a random variable $X = (X_1, \ldots, X_d)^\top$, with absolute continuous distribution function $F$ and copula $C_X$, the density $c_X$ is obtained by differen-

tiating (17.6):

$$c_X(u_1, \ldots, u_d) = \frac{f\{F_{X_1}^{-1}(u_1), \ldots, F_{X_d}^{-1}(u_d)\}}{\prod_{j=1}^{d} f_j\{F_{X_j}^{-1}(u_j)\}} \tag{17.8}$$

where $f$ is the joint density of $F_X$ and $f_j$ the density of $F_{X_j}$. The density from the copula of $X$ can be determined from its joint density and inverse marginal distributions.

### 17.1.1 Gaussian Copula

The Gaussian copula represents the *dependence structure* of the multivariate normal distribution, that means, *normal* marginal distributions are combined with a Gaussian copula to form multivariate normal distributions. The combination of *non-normal* marginal distributions with a Gaussian copula results in *meta-Gaussian* distributions, i.e., distributions where *only* the dependence structure is Gaussian.

To obtain the Gaussian copula, let $X = (X_1, \ldots, X_d)^\top \sim N_d(\mu, \Sigma)$ with $X_j \sim N(\mu_j, \sigma_j)$ for $j = 1, \ldots, d$. From Sklar's Theorem there exists a copula $C_X$ such that:

$$F_X(x_1, \ldots, x_d) = C_X\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\}$$

where $F_{X_j}$ is the distribution function of $X_j$ and $F_X$ the distribution function of $X$.

Let $Y_j = T_j(X_j)$, where $T_j(x)$ is the transformation

$$T_j(x) = \frac{x - \mu_j}{\sigma_j}.$$

Then $Y_j \sim N(0, 1)$ and $Y = (Y_1, \ldots, Y_d)^\top \sim N_d(0, \Psi)$ where $\Psi$ is the correlation matrix associated with $\Sigma$. Moreover, a copula $C_\Psi^{Ga}$, called *Gaussian copula* exists as follows:

$$F_Y(y_1, \ldots, y_d) = C_\Psi^{Ga}\{\Phi(y_1), \ldots, \Phi(y_d)\} \tag{17.9}$$

where $\Phi$ is the standard Normal cdf of $Y_j$ and $F_Y$ the distribution function of $Y$. An explicit expression for the Gaussian copula is obtained by rewriting (17.9) with $u_j = \Phi(y_j)$:

$$C_\Psi^{Ga}(u_1, \ldots, u_d) = F_Y\{\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\}$$

$$= \int_{-\infty}^{\Phi^{-1}(u_1)} \dots \int_{-\infty}^{\Phi^{-1}(u_d)} (2\pi)^{-\frac{d}{2}} \mid \Psi \mid^{-\frac{1}{2}} \exp\left(-\frac{1}{2} r^\top \Psi^{-1} r\right) dr_1 \dots dr_d.$$

As $T_j(x) = \frac{x-\mu_j}{\sigma_j}$ is increasing, it follows from Theorem 17.2 that

$$C_X = C_\Psi^{Ga}.$$

Thus, *any* multivariate normal distribution can be constructed from its marginal distributions and the Gaussian copula $C_\Psi^{Ga}$ with the desired correlation matrix $\Psi$.

**Remark 17.1**
*If $\Psi = I_d$ the Gaussian copula becomes the product copula as*

$$\begin{aligned} C_{I_d}^{Ga}(u_1, \dots, u_d) &= \int_{-\infty}^{\Phi^{-1}(u_1)} \dots \int_{-\infty}^{\Phi^{-1}(u_d)} (2\pi)^{-\frac{d}{2}} \exp\left(-\frac{1}{2} \sum_{j=1}^{d} r_j^2\right) dr_1 \dots dr_d \\ &= \int_{-\infty}^{\Phi^{-1}(u_1)} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2} r_1^2) dr_1 \dots \int_{-\infty}^{\Phi^{-1}(u_d)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} r_d^2\right) dr_d \\ &= \Phi\{\Phi^{-1}(u_1)\} \dots \Phi\{\Phi^{-1}(u_d)\} \\ &= \Pi(u_1, \dots, u_d). \end{aligned}$$

The *density of the Gaussian copula* (Figure 17.2) is obtained by differentiating (17.9),

$$\begin{aligned} \mid 2\pi\Psi \mid^{-\frac{1}{2}} \exp\left(-\frac{1}{2} x^\top \Psi^{-1} x\right) &= c_\Psi^{Ga}\{\Phi(x_1), \dots, \Phi(x_d)\} \\ &\quad \times \prod_{j=1}^{d} (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} x_j^2\right) \end{aligned}$$

rearranging terms and defining $\zeta_j = \Phi^{-1}(u_j)$, $\zeta = (\zeta_1, \dots, \zeta_d)^\top$:

$$c_\Psi^{Ga}(u_1, \dots, u_d) = \mid \Psi \mid^{-\frac{1}{2}} \exp\left\{-\frac{1}{2} \zeta^\top (\Psi^{-1} - I_d) \zeta\right\}. \tag{17.10}$$

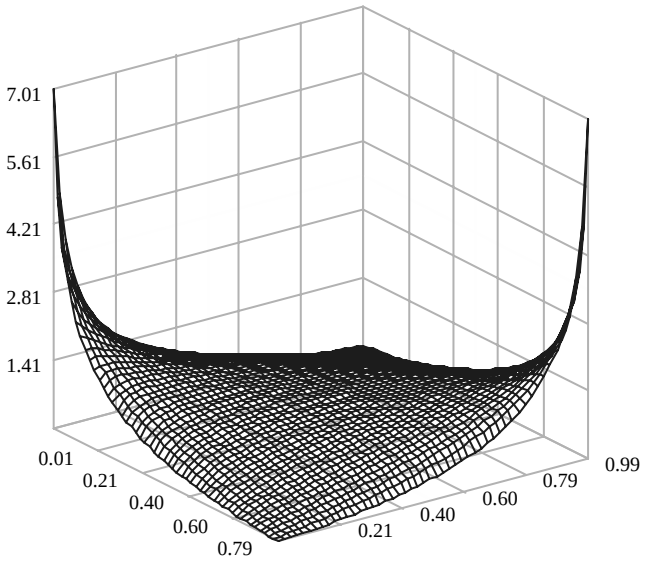


Figure 17.2: Density of the Gaussian copula, $c_{\Psi}^{Ga}(u_1, u_2)$, $\psi_{12} = 0.5$.
SFEgausscop

### 17.1.2 Student's $t$-Copula

The $t$-copula, containing the dependence structure form the multivariate $t$-distribution, may be obtained in a similar way.

Let $X = (X_1, \ldots, X_d)^\top \sim t_d(\nu, \mu, \Sigma)$ and $Y = (Y_1, \ldots, Y_d)^\top \sim t_d(\nu, 0, \Psi)$ where $\Psi$ is the correlation matrix associated with $\Sigma$. The unique copula from $Y$ is the *Student's t-copula* $C_{\nu,\Psi}^t$. Moreover, it follows from Theorem 17.2 that $C_X = C_{\nu,\Psi}^t$.

For $u = (u_1, \ldots, u_d)^\top \in [0,1]^d$, the *Student's t-copula* is given by

$$C_{\nu,\Psi}^t(u_1, \ldots, u_d) = t_{\nu,\Psi}\{t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d)\}$$

where $t_\nu^{-1}$ is the quantile function from the univariate $t$-distribution and $t_{\nu,\Psi}$ the distribution function of $Y$.

The *density of the t-copula* (Figure 17.3) is given by

$$c_{\nu,\Psi}^t(u_1, \ldots, u_d) = \frac{t_{\nu,\Psi}\{t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d)\}}{\prod_{j=1}^d t_{\nu,\Psi}\{t_\nu^{-1}(u_j)\}}. \tag{17.11}$$

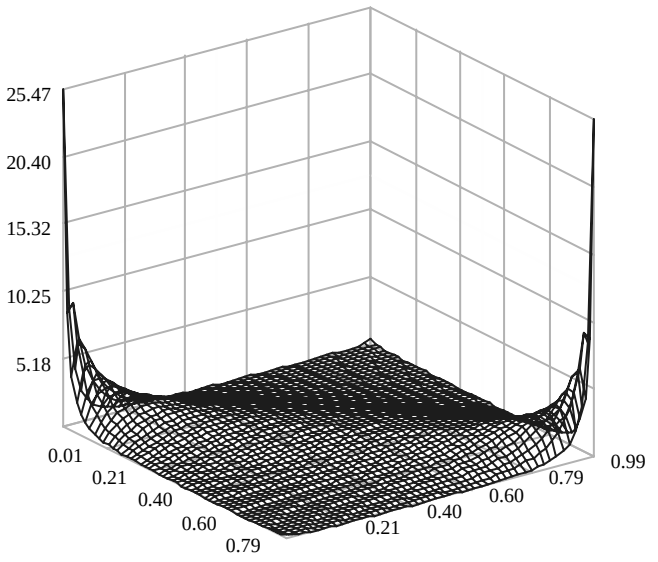


Figure 17.3: Density of $t$-copula, $c^t_{\nu,\Psi}(u_1, u_2)$, $\psi_{12} = 0.2$, $\nu = 3$.
SFEtcop

With $\zeta_j = t_\nu^{-1}(u_j)$ the density of the $t$-copula can be expressed as:

$$c^t_{\nu,\Psi}(u_1, \ldots, u_d) = \mid \Psi \mid^{-\frac{1}{2}} \frac{\Gamma(\frac{\nu+d}{2}) \left\{\Gamma(\frac{\nu}{2})\right\}^{d-1} \left(1 + \frac{1}{\nu}\zeta^\top \Psi^{-1}\zeta\right)^{-\frac{\nu+d}{2}}}{\left\{\Gamma(\frac{\nu+1}{2})\right\}^{d} \prod_{j=1}^{d} \left(1 + \frac{1}{\nu}\zeta_j^2\right)^{-\frac{\nu+1}{2}}}. \quad (17.12)$$

### 17.1.3 Archimedean Copulae

**Definition 17.4**
*Let $\phi : [0,1] \to [0,\infty]$ be a continuous, strictly decreasing function with $\phi(1) = 0$. The pseudo inverse of $\phi$ is the function $\phi^{[-1]}$ such that*

$$\phi^{[-1]} = \begin{cases} \phi^{-1}(t), & 0 \le t \le \phi(0) \\ 0, & \phi(0) \le t \le \infty \end{cases}$$

**Theorem 17.3**
*Let $\phi : [0,1] \to [0,\infty]$ be a convex, strictly decreasing continuous function*

*with $\phi(1) = 0$. Then the function $C : [0,1]^2 \to [0,1]$*

$$C(u_1, u_2) = \phi^{[-1]}\{\phi(u_1) + \phi(u_2)\} \tag{17.13}$$

*is a copula.*

Copulae of the form (17.13) are called *Archimedean copulae* and the functions $\phi$ their *generators*. If in addition $\phi(0) = \infty$, $\phi$ is called a *strict generator* and $\phi^{[-1]} = \phi^{-1}$.

**Example 17.1 (Gumbel copula)**
*The function $\phi(t) = (-\log t)^\theta$, $\theta \in [1, \infty)$ is convex, strictly decreasing and continuous in $[0,1]$ with $\phi(0) = \infty$ and $\phi(1) = 0$, thus it is a strict generator and $\phi^{-1}(t) = e^{-t\theta^{-1}}$. The function $C : [0,1]^2 \to [0,1]$*

$$C(u_1, u_2) = \exp\left[-\{(-\log u_1)^\theta + (-\log u_2)^\theta\}^{\theta^{-1}}\right]$$

*is the Gumbel copula . For $\theta = 1$ we obtain the product copula: $C(u_1, u_2) = \Pi(u_1, u_2)$, for $\theta \to \infty$ we obtain the Fréchet-Hoeffding upper bound:*

$$C_\theta(u_1, u_2) \longrightarrow \min(u_1, u_2) \quad as \quad \theta \to \infty.$$

### 17.1.4 Multivariate Archimedean Copulae

The next theorem generalizes the concepts of Archimedean copulae for the $d$-dimensional case.

**Definition 17.5**
*A function $f(t)$ is completely monotonic in an interval $[a,b]$ if for $t \in [a,b]$ and $k \in \mathbb{N}$ it satisfies*

$$(-1)^k \frac{d^k}{dt^k} f(t) \geq 0$$

**Theorem 17.4**
*Let $\phi$ be a strict generator. The function $C^d : [0,1]^d \to [0,1]$*

$$C^d(u_1, \ldots, u_d) = \phi^{-1}\{\phi(u_1) + \ldots + \phi(u_d)\}$$

*is a copula for all $d \geq 2$ if and only if $\phi^{-1}$ is completely monotonic in $[0, \infty)$.*

Some $d$-dimensional Archimedean copulae are presented below.

1. **Frank copula**, $0 < \theta \leq \infty$

$$C_\theta(u_1, \ldots, u_d) = -\frac{1}{\theta} \log \left[ 1 + \frac{\prod_{j=1}^{d} \{\exp(-\theta u_j) - 1\}}{\{\exp(-\theta) - 1\}^{d-1}} \right].$$

   The dependence becomes maximal when $\theta$ tends to infinity and independence is achieved when $\theta = 0$.

2. **Gumbel copula**, $1 \leq \theta \leq \infty$

$$C_\theta(u_1, \ldots, u_d) = \exp \left[ - \left\{ \sum_{j=1}^{d} (-\log u_j)^\theta \right\}^{\theta^{-1}} \right].$$

   For $\theta > 1$ this copula allows for the generation of dependence in the upper tail. For $\theta = 1$, the Gumbel copula reduces to the product copula, i.e.

$$C_1(u_1, \ldots, u_d) = \prod_{j=1}^{d} u_j.$$

   For $\theta \to \infty$we obtain the Fréchet-Hoeffding upper bound:

$$C_\infty(u_1, \ldots, u_d) = \min(u_1, \ldots, u_d).$$

3. **Ali-Mikhail-Haq copula**, $-1 \leq \theta < 1$

$$C_\theta(u_1, \ldots, u_d) = \frac{\prod_{j=1}^{d} u_j}{1 - \theta \left\{ \prod_{j=1}^{d} (1 - u_j) \right\}}.$$

   If $\theta = 0$, then we have independence:

$$C_0(u_1, \ldots, u_d) = \prod_{j=1}^{d} u_j.$$

4. **Clayton copula**, $\theta > 0$

$$C_\theta(u_1, \ldots, u_d) = \left\{ \left( \sum_{j=1}^{d} u_j^{-\theta} \right) - d + 1 \right\}^{-\theta^{-1}}$$

where the density of the Clayton copula is given by

$$c_\theta(u_1, \ldots, u_d) = \prod_{j=1}^{d} \{1 + (j-1)\theta\} u_j^{-(\theta+1)} \left( \sum_{j=1}^{d} u_j^{-\theta} - d + 1 \right)^{-(\theta^{-1}+d)}.$$

As the parameter $\theta$ tends to infinity, dependence becomes maximal and as $\theta$ tends to zero, we have independence. As $\theta \to 1$, the distribution tends to the lower Fréchet bound. The Clayton copula allows for the generation of asymmetric dependence and lower tail dependence, but no upper tail dependence.

### 17.1.5 Distributions Constructed with Copulae

Joint distributions with different dependence between the marginal distributions can be easily constructed with copulae. As an example, the standard normal and $t_3$ marginal distributions are coupled with 4 distinct copulae $C$ to form the joint distribution $F$ given by

$$F(x_1, x_2) = C\{\Phi(x_1), t_3(x_2)\}.$$

The density function of $F$ is

$$f(x_1, x_2) = c\{\Phi(x_1), t_3(x_2)\}\varphi(x_1) f_{t,3}(x_2)$$

where $\varphi(x)$ is the density function from the standard normal distribution and $f_{t,3}(x)$ from the $t$-distribution with 3 degrees of freedom. The contour plots from $f(x_1, x_2)$ are shown in Figure 17.4 for the respective copula choices.

### 17.1.6 Monte Carlo Simulation

The simulation from $d$ pseudo random variables with joint distribution defined by a copula $C$ and $d$ marginal distributions $F_j$, $j = 1, \ldots, d$, may follow different techniques.

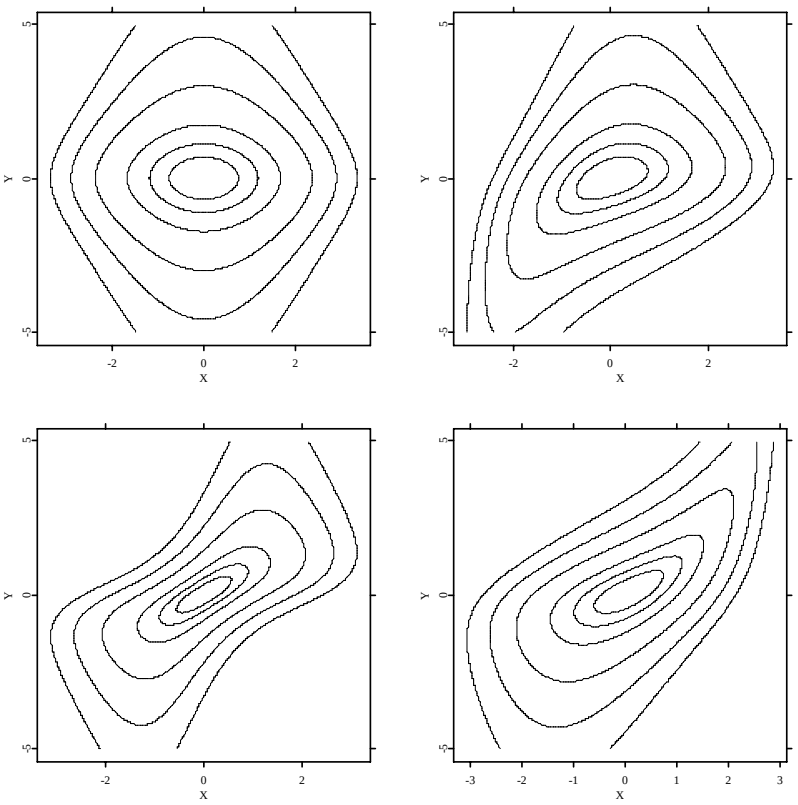


Figure 17.4: pdf contour plots, $F(x_1, x_2) = C\{\Phi(x_1), t_3(x_2)\}$ with (clockwise) Gaussian ($\rho = 0$), Clayton ($\theta = 0.9$), Frank ($\theta = 8$) and Gumbel ($\theta = 2$) copulae.

SFEplotCop

Defining the copula $j$-dimensional marginal distribution $C_j$ for $j = 2, \ldots, d-1$ as

$$C_j(u_1, \ldots, u_j) = C(u_1, \ldots, u_j, 1, \ldots, 1)$$

and the derivative of $C_j$ with respect to the first $j-1$ arguments as

$$c^j_{j-1}(u_1, \ldots, u_j) = \frac{\partial^{j-1} C_j(u_1, \ldots, u_j)}{\partial u_1, \ldots, \partial u_{j-1}}$$

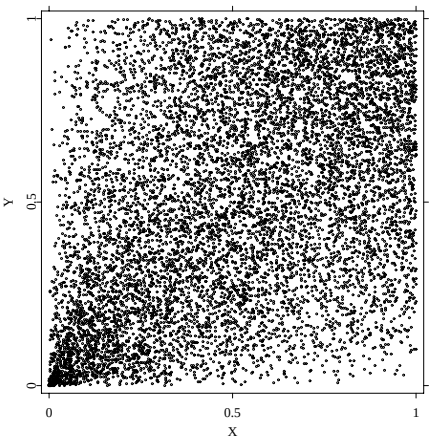


Figure 17.5: Monte Carlo sample of 10.000 realisations of pseudo random variable with uniform marginals in $[0, 1]$ and dependence structure given by Clayton copula, $\theta = 0.79$.

SFEclaytonMC

the probability $P(U_j \leq u_j, U_1 = u_1, \ldots, U_{j-1} = u_{j-1})$ can be written as

$$\lim_{\Delta u_1, \ldots, \Delta u_{j-1} \to 0} \frac{C_j(u_1 + \Delta u_1, \ldots, u_{j-1} + \Delta u_{j-1}, u_j) - C_j(u_1, \ldots, u_j)}{\Delta u_1, \ldots, \Delta u_{j-1}}$$

$$= c_{j-1}^j(u_1, \ldots, u_j).$$

Thus, the conditional distribution $\Lambda(u_j)$ (given fixed $u_1, \ldots, u_{j-1}$) is a function of the ratio of derivatives:

$$\begin{aligned} \Lambda(u_j) &= P(U_j \leq u_j \mid U_1 = u_1, \ldots, U_{j-1} = u_{j-1}) \\ &= \frac{P(U_j \leq u_j, U_1 = u_1, \ldots, U_{j-1} = u_{j-1})}{P(U_1 = u_1, \ldots, U_{j-1} = u_{j-1})} \\ &= \frac{c_{j-1}^j(u_1, \ldots, u_j)}{c_{j-1}^{j-1}(u_1, \ldots, u_{j-1})}. \end{aligned}$$

The generation of $d$ pseudo random numbers with given marginal distributions $F_j$, $j = 1, \ldots, d$ and dependence structure given by the copula $C$ follows the steps:

1. generate pseudo random numbers $v_1, \ldots, v_d$ independent and uniformly distributed in $[0, 1]$.
2. for $j = 1, \ldots, d$ generate the pseudo random numbers as $u_j = \Lambda^{-1}(v_j)$. The pseudo random numbers $u_1, \ldots, u_d$ have uniform marginal distributions in $[0, 1]$ and dependence structure given by the copula $C$ (Figure 17.5).
3. set $x_j = F_j^{-1}(u_j)$. The pseudo random numbers $x_1, \ldots, x_d$ are distributed with the desired marginal distributions and dependence structure (Figure 17.6).

If $C$ is the Gaussian copula, the simulation is as follows:

1. generate pseudo random numbers $v_1, \ldots, v_d$ distributed as $N(0, \Psi)$
2. set $u_j = \Phi(v_j)$, $j = 1, \ldots, d$. The pseudo random numbers $u = (u_1, \ldots, u_d)$ have uniform marginal distributions in $[0, 1]$ and dependence structure given by $C_\Psi^{Ga}$.
3. set $x_j = F_j^{-1}(u_j)$. The pseudo random numbers $x_1, \ldots, x_d$ are distributed with the desired marginal distributions and dependence structure.

If the marginal distributions are normal, the pseudo random numbers are multivariate normal distributed. Otherwise their distribution is called *Meta-Gaussian* distribution.

If $C$ is the $t$-copula, the simulation is as follows:

1. generate pseudo random numbers $v_1, \ldots, v_d$ distributed as $t_d(\nu, 0, \Psi)$
2. set $u_j = t_\nu(v_j)$, $j = 1, \ldots, d$ where $t_\nu$ is the univariate $t$ distribution with $\nu$ degrees of freedom. The pseudo random numbers $u = (u_1, \ldots, u_d)$ have uniform marginal distributions in $[0, 1]$ and dependence structure given by $C_{\nu,\Psi}^t$.
3. set $x_j = F_j^{-1}(u_j)$. The pseudo random numbers $x_1, \ldots, x_d$ are distributed with the desired marginal distributions and dependence structure.

If the marginal distributions are $t_\nu$, the pseudo random numbers are multivariate $t$ distributed. Otherwise their distribution is called *Meta-t* distribution.

Repeating one of the procedures above $T$ times yields a Monte Carlo sample $\{x_{j,t}\}_{t=1}^T$, for $j = 1, \ldots, d$ of a random variable distributed as desired.

Scatterplots of Monte Carlo sample of pseudo random variable with uniform and $t_3$ marginal distributions and dependence structure given by $t$-copula are

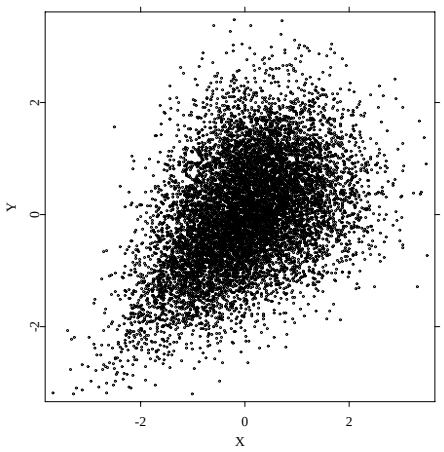


Figure 17.6: Monte Carlo sample of 10.000 realisations of pseudo random variable with standard normal marginals and dependence structure given by Clayton copula with $\theta = 0.79$.
SFEclaytonMC

plotted in Figure 17.7.

## 17.2 Copula Estimation

Let $X$ be a $d$-dimensional random variable with parametric univariate marginal distributions $F_{X_j}(x_j; \delta_j)$, $j = 1, \ldots, d$. Further let a copula belong to a parametric family $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$. From Sklar's Theorem the distribution of $X$ can be expressed as

$$F_X(x_1, \ldots, x_d) = C\{F_{X_1}(x_1; \delta_1), \ldots, F_{X_d}(x_d; \delta_d); \theta\}$$

and its density as

$$f(x_1, \ldots, x_d; \delta_1, \ldots, \delta_d, \theta) = c\{F_{X_1}(x_1; \delta_1), \ldots, F_{X_d}(x_d; \delta_d); \theta\} \prod_{j=1}^{d} f_j(x_j; \delta_j)$$

where

$$c(u_1, \ldots, u_d) = \frac{\partial^d C(u_1, \ldots, u_d)}{\partial u_1 \ldots \partial u_d}.$$

For a sample of observations $\{x_t\}_{t=1}^T$, $x_t = (x_{1,t}, \ldots, x_{d,t})^\top$ and a vector of

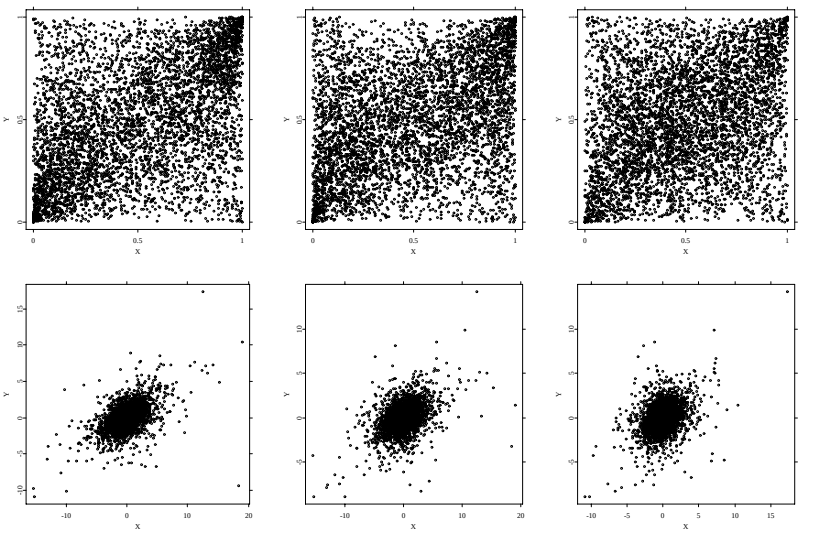

Figure 17.7: Scatterplots of Monte Carlo sample (5.000 realisations) of pseudo random variable $X = (X_1, X_2, X_3)^\top$ with uniform (above) and $t_3$ marginal distributions (below). Dependence structure given by $t$-copula with $\nu = 3$ and $\psi_{i,j} = 0.5$, $i, j = 1, 2, 3$, $i \neq j$.

SFEtMC

parameters $\alpha = (\delta_1, \ldots, \delta_d, \theta)^\top \in \mathbb{R}^{k+1}$ the likelihood function is given by

$$L(\alpha; x_1, \ldots, x_T) = \prod_{t=1}^{T} f(x_{1,t}, \ldots, x_{d,t}; \delta_1, \ldots, \delta_d, \theta)$$

and the log-likelihood function by

$$\ell(\alpha; x_1, \ldots, x_T) = \sum_{t=1}^{T} \log c\{F_{X_1}(x_{1,t}; \delta_1), \ldots, F_{X_d}(x_{d,t}; \delta_d); \theta\}$$

$$+ \sum_{t=1}^{T} \sum_{j=1}^{d} \log f_j(x_{j,t}; \delta_j).$$

The vector of parameters $\alpha = (\delta_1, \ldots, \delta_d, \theta)^\top$ contains $d$ parameters $\delta_j$ from the marginals and the copula parameter $\theta$. All these parameters can be estimated *in one step*. For practical applications, however, a two step estimation procedure is more efficient.

### 17.2.1 Maximum Likelihood Estimation

In the Maximum Likelihood estimation method (also called *full maximum likelihood*), the vector of parameters $\alpha$ is estimated in one single step through

$$\tilde{\alpha}_{FML} = \arg\max_{\alpha} \ell(\alpha)$$

The estimates $\tilde{\alpha}_{FML} = (\tilde{\delta}_1, \ldots, \tilde{\delta}_d, \tilde{\theta})^\top$ solve

$$(\partial\ell/\partial\delta_1, \ldots, \partial\ell/\partial\delta_d, \partial\ell/\partial\theta) = 0.$$

### 17.2.2 IFM - Inference for Margins

In the IFM (*inference for margins*) method, the parameters $\delta_j$ from the marginal distributions are estimated in the first step and used to estimate the dependece parameter $\theta$ in the second step:

1. for $j = 1, \ldots, d$ the log-likelihood function for each of the marginal distributions are
$$\ell_j(\delta_j) = \sum_{t=1}^{T} \log f_j(x_{j,t}; \delta_j)$$
and the estimated parameters
$$\hat{\delta}_j = \arg\max_{\delta} \ell_j(\delta_j)$$

2. the *pseudo log-likelihood* function
$$\ell(\theta, \hat{\delta}_1, \ldots, \hat{\delta}_d) = \sum_{t=1}^{T} \log c\{F_{X_1}(x_{1,t}; \hat{\delta}_1), \ldots, F_{X_d}(x_{d,t}; \hat{\delta}_d); \theta\}$$
is maximised over $\theta$ to get the dependence parameter estimate $\hat{\theta}$.

The estimates $\hat{\alpha}_{IFM} = (\hat{\delta}_1, \ldots, \hat{\delta}_d, \hat{\theta})^\top$ solve

$$(\partial\ell_1/\partial\delta_1, \ldots, \partial\ell_d/\partial\delta_d, \partial\ell/\partial\theta) = 0.$$

### 17.2.3 CML - Canonical Maximum Likelihood

In the CML (*canonical maximum likelihood*) method, the univariate marginal distributions are estimated through the empirical distribution function $\hat{F}$.

For $j = 1, \ldots, d$

$$\hat{F}_j(x) = \frac{1}{T+1} \sum_{t=1}^{T} \mathbf{1}(x_{j,t} \leq x).$$

The *pseudo log-likelihood* function is

$$\ell(\theta) = \sum_{t=1}^{T} \log c\{\hat{F}_1(x_{1,t}), \ldots, \hat{F}_d(x_{d,t}); \theta\}$$

and the copula parameter estimator $\hat{\theta}_{CML}$ is given by

$$\hat{\theta}_{CML} = \arg\max_{\theta} \ell(\theta).$$

Notice that the first step of the IMF and CML methods estimates the marginal distributions. After marginals are estimated, a *pseudo sample* $\{u_t\}$ of observations transformed in the unit $d$-cube is obtained and used in the *copula* estimation.

### 17.2.4 Gaussian Copula Estimation

From a sample $\{u_t\}_{t=1}^{T}$ where $u = (u_1, \ldots, u_d)^\top \in [0,1]^d$, the density of the Gaussian copula is given by

$$c_{\Psi}^{Ga}(u_1, \ldots, u_d) = \mid \Psi \mid^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\zeta^\top(\Psi^{-1} - I_d)\zeta\right\}$$

and the pseudo log-likelihood function by

$$\ell(\Psi; u_{1,t}, \ldots, u_{d,t}) = -\frac{T}{2} \log \mid \Psi \mid -\frac{1}{2} \sum_{t=1}^{T} \zeta_t^\top(\Psi^{-1} - I_d)\zeta_t$$

where $\zeta_t = (\zeta_{1,t}, \ldots, \zeta_{d,t})^\top$ and $\zeta_{j,t} = \Phi^{-1}(u_{j,t})$.

The maximum-likelihood estimator for $\Psi$ is

$$\hat{\Psi} = \arg\max_{\Psi \in \mathcal{P}} \ell(\Psi)$$

where $\mathcal{P}$ is the set of all lower-triangular matrices with one in the diagonal. The maximisation is feasible but very slow for high dimensions, see Embrechts, Frey and McNeil (2005). An approximate solution can be obtained

using the ML estimator for the covariance matrix $\Sigma$ as

$$\hat{\Sigma} = \arg\max_{\Sigma} \ell(\Sigma).$$

The estimator is then

$$\hat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T} \zeta_t \zeta_t^\top$$

and defining

$$\Lambda = \operatorname{diag}(\hat{\Sigma}_{ii})$$

we obtain

$$\hat{\Psi} = \Lambda^{-1}\hat{\Sigma}\Lambda^{-1}.$$

### 17.2.5 $t$-Copula Estimation

One possible estimation method for the Student's $t$-copula is based on the estimation of Kendall's tau with method of moments, as in Embrechts (2005). For a pseudo sample $\{u_t\}_{t=1}^{T}$ where $u = (u_1, \ldots, u_d)^\top \in [0,1]^d$, the Kendall's tau coefficient for each pair of observations $i, j = 1, \ldots, d$ is given by

$$\hat{\rho}_\tau(u_i, u_j) = \binom{T}{2}^{-1} \sum_{1 \le t_1 \le t_2 \le T} \operatorname{sign}(u_{i,t_1} - u_{i,t_2})(u_{j,t_1} - u_{j,t_2}).$$

Each element from the correlation matrix $\Psi$ is estimated as

$$\hat{\psi}_{ij} = \sin\left\{\frac{\pi}{2}\hat{\rho}_\tau(u_i, u_j)\right\}.$$

The parameter $\nu$ is estimated through maximum-likelihood with the estimated matrix $\hat{\Psi}$ held fixed. In this case the pseudo log-likelihood function is given by

$$\ell(\nu; u_{1,t}, \ldots, u_{d,t}) = \sum_{t=1}^{T} \log\{c^t_{\nu,\hat{\Psi}}(u_{1,t}, \ldots, u_{d,t})\}$$

where $c^t_{\nu,\hat{\Psi}}(u_{1,t}, \ldots, u_{d,t})$ is defined in equation (17.11). The estimator for the number of degrees of freedom is then

$$\hat{\nu} = \arg\max_{\nu \in \mathbb{N}_+} \ell(\nu).$$

## 17.3 Value-at-Risk and Copulae

This section introduces the main assumptions and steps necessary for estimating the VaR from a linear portfolio using copulae. Static and time-varying methods as well as their VaR performance evaluation through backtesting are described below.

### 17.3.1 Value-at-Risk

Let $w = (w_1, \ldots, w_d)^\top \in \mathbb{R}^d$ denote a portfolio of positions on $d$ assets and $S_t = (S_{1,t}, \ldots, S_{d,t})^\top$ be a non-negative random vector representing the prices of the assets at $t$, where $t$ is a time index. The value $V_t$ of the portfolio $w$ is given by

$$V_t = \sum_{j=1}^{d} w_j S_{j,t} \tag{17.14}$$

and the random variable

$$L_{t+\tau} = (V_{t+\tau} - V_t) \tag{17.15}$$

also called *profit and loss (P&L) function*, expresses the change in the portfolio value between $\tau$ periods.

Defining the *log-returns* $X_{t+\tau}$ in $\tau$ periods as $X_{t+\tau} = \log S_{t+\tau} - \log S_t$ and considering $\tau = 1$, (17.15) can be written as

$$L_{t+1} = \sum_{j=1}^{d} w_j S_{j,t} \left\{ \exp(X_{j,t+1}) - 1 \right\}. \tag{17.16}$$

The distribution function from $L$, dropping the time index, is given by

$$F_L(x) = P(L \leq x). \tag{17.17}$$

The *Value-at-Risk* at level $\alpha$ from a portfolio $w$ is defined as the $\alpha$-quantile from $F_L$:

$$VaR(\alpha) = F_L^{-1}(\alpha). \tag{17.18}$$

It follows from (17.16) and (17.17) that $F_L$ depends on the $d$-dimensional distribution of log-returns $F_X$. In general, the *loss distribution* $F_L$ depends on a random process representing the *risk factors* influencing the P&L from a portfolio. In the present case log-returns are a suitable risk factor choice.

Thus, modelling their distribution is essential to obtain the quantiles from $F_L$.

A log-returns process $\{X_t\}$ can be modelled as

$$X_{j,t} = \mu_{j,t} + \sigma_{j,t}\varepsilon_{j,t}$$

where $\varepsilon_t = (\varepsilon_{1,t}, \ldots, \varepsilon_{d,t})^\top$ are standardised *i.i.d.* innovations with $E[\varepsilon_{j,t}] = 0$ and $E[\varepsilon_{j,t}^2] = 1$ for $j = 1, \ldots, d$; $\mathcal{F}_t$ is the available information at time $t$;

$$\mu_{j,t} = E[X_{j,t} \mid \mathcal{F}_{t-1}]$$

is the conditional mean given $\mathcal{F}_{t-1}$ and

$$\sigma_{j,t}^2 = E[(X_{j,t} - \mu_{j,t})^2 \mid \mathcal{F}_{t-1}]$$

is the conditional variance given $\mathcal{F}_{t-1}$. The innovations $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d)^\top$ have joint distribution $F_\varepsilon$ and $\varepsilon_j$ has continuous marginal distributions $F_j$, $j = 1, \ldots, d$.

### 17.3.2 VaR Estimation with Copulae

The innovations $\varepsilon$ have distribution function described by

$$F_\varepsilon(\varepsilon_1, \ldots, \varepsilon_d) = C_\theta\{F_1(\varepsilon_1), \ldots, F_d(\varepsilon_d)\} \tag{17.19}$$

where $C_\theta$ is a copula belonging to a parametric family $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$. To obtain the Value-at-Risk in this set up, the dependence parameter and distribution function from residuals are estimated from a sample of log-returns and used to generate P&L Monte Carlo samples. Their quantiles at different levels are the estimators for the Value-at-Risk. The whole procedure is summarised below.

For a portfolio $w$ on $d$ assets and a sample $\{x_{j,t}\}_{t=1}^T$, $j = 1, \ldots, d$ of log-returns, the Value-at-Risk at level $\alpha$ is estimated according to the following steps:

1. estimation of residuals $\hat{\varepsilon}_t$
2. specification and estimation of marginal distributions $F_j(\hat{\varepsilon}_j)$
3. specification of a parametric copula family $\mathcal{C}$ and estimation of dependence parameter $\theta$
4. generation of Monte Carlo sample of innovations $\varepsilon$ and losses $L$
5. estimation of $\widehat{VaR}(\alpha)$, the empirical $\alpha$-quantile from $L$.

### 17.3.3 Time-Varying Copulae and Backtesting

The application of the (*static*) procedure described above on sliding windows of time series $\{x_{j,t}\}_{t=1}^{T}$ delivers a sequence of parameters for a copula family. Hence the denomination *time-varying copulae*.

Using subsets of size $w$ scrolling in time $t$ (i.e., a *moving window* of size $w$),

$$\{x_t\}_{t=s-w+1}^{s}$$

for $s = w, \ldots, T$, the procedure described in Section 17.3.2 generates the time series $\{\widehat{VaR}_t\}_{t=w}^{T}$ of Value-at-Risk and $\{\hat{\theta}_t\}_{t=w}^{T}$ dependence parameters estimates.

*Backtesting* is used to evaluate the performance of the specified copula family $\mathcal{C}$. The estimated values for the VaR are compared with the true realisations $\{l_t\}$ of the P&L function, an *exceedance* occuring for each $l_t$ smaller than $\widehat{VaR}_t(\alpha)$. The ratio of the number of exceedances to the number of observations gives the *exceedances ratio* $\hat{\alpha}$:

$$\hat{\alpha} = \frac{1}{T-w} \sum_{t=w}^{T} \mathbf{1}\{l_t < \widehat{VaR}_t(\alpha)\}.$$

## 17.4 Empirical Results

The estimation methods described in the preceeding section are used on two exchange rates portfolio, the first composed of 2 positions, the second of 5 positions. Different copulae are used in static and dynamic set up and their VaR performance is compared based on backtesting.

### 17.4.1 An Exchange Rate Portfolio

In this section, the Value-at-Risk of portfolios on exchange rates (DEM/USD and GBP/USD from 01.12.1979 to 01.04.1994) is computed using different copulae. Assuming the log-returns $\{X_{j,t}\}$ follow a GARCH(1,1) process we have

$$X_{j,t} = \mu_{j,t} + \sigma_{j,t}\varepsilon_{j,t}$$

where

$$\sigma_{j,t}^2 = \omega_j + \alpha_j \sigma_{j,t-1}^2 + \beta_j (X_{j,t-1} - \mu_{j,t-1})^2$$

| | $\hat{\omega}_j$ | $\hat{\alpha}_j$ | $\hat{\beta}_j$ |
|---|---|---|---|
| $j = 1$ | 0.00 | 0.07 | 0.89 |
| $j = 2$ | 0.00 | 0.05 | 0.93 |

Table 17.1: GARCH(1,1) parameters, 2-dimensional portfolio.

| | $\hat{\mu}_j$ | $\hat{\sigma}_j$ |
|---|---|---|
| $j = 1$ | 0.0081 | 0.9987 |
| $j = 2$ | 0.1887 | 0.9991 |

Table 17.2: Parameters from marginal distributions.

and $\omega > 0$, $\alpha_j \geq 0$, $\beta_j \geq 0$, $\alpha_j + \beta_j < 1$.

The fit of a GARCH(1,1) model to the sample of log returns $\{x_t\}_{t=1}^T$, $x_t = (x_{1,t}, x_{2,t})^\top$, $T = 3718$, gives the estimates $\hat{\omega}_j$, $\hat{\alpha}_j$ and $\hat{\beta}_j$, as in Table 17.1, and empirical residuals $\{\hat{\varepsilon}_t\}_{t=1}^T$, where $\hat{\varepsilon}_t = (\hat{\varepsilon}_{1,t}, \hat{\varepsilon}_{2,t})^\top$. The scatterplot of the empirical residuals is depicted in Figure 17.8.

The marginal distributions are specified as normal, i.e., $\hat{\varepsilon}_j \sim N(\hat{\mu}_j, \hat{\sigma}_j)$ with parameters $\hat{\delta}_j = (\hat{\mu}_j, \hat{\sigma}_j)$ estimated from the data as in Table 17.2.

Figure 17.9 displays the Kernel density estimator of the residuals and of the normal density, estimated with a Quartic kernel.

### Static Copulae

The dependence parameters are estimated (Table 17.3) for different copula families (Gaussian, Clayton and Gumbel). Various portfolios are used to generate the P&L samples and the estimated Value-at-Risk for each of them are in Table 17.4. Residuals $\hat{\varepsilon}$ and fitted copulae (Gaussian, Clayton and Gumbel) are plotted in Figure 17.10.

| Copula | $\hat{\theta}$ |
|---|---|
| Gaussian | 0.767 |
| Clayton | 1.861 |
| Gumbel | 2.283 |

Table 17.3: Dependence parameter for different static copulae.

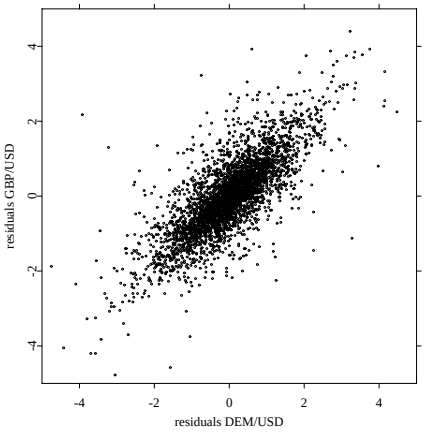


Figure 17.8: Scatterplot from residuals $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$.
SFEresGarch

### Time-varying Copulae

In the dynamic approach, the empirical residuals are sampled in moving windows with fixed size $w = 250$, $\{\hat{\varepsilon}_t\}_{t=s-w+1}^{s}$, for $s = w, \ldots, T$. The time series from estimated dependence parameters for each copula family are in Figure 17.11.

The same portfolio compositions as in the static case are used to generate P&L samples. The series of estimated Value-at-Risk and the P&L function for selected portfolios are plotted in Figure 17.12, 17.13. and 17.14. Backtest-

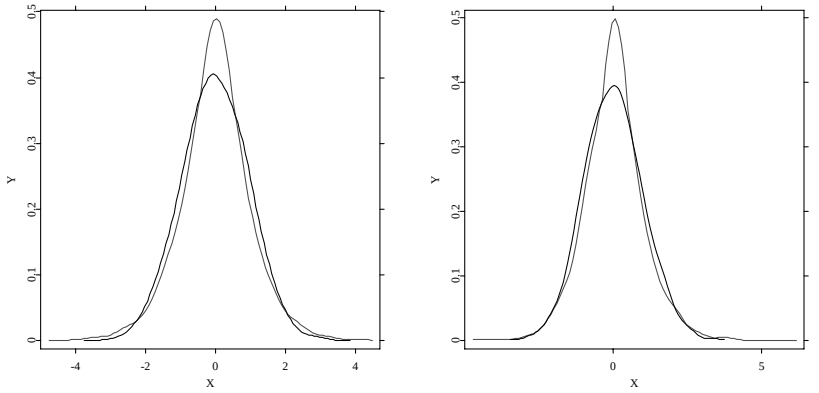


Figure 17.9: Kernel density estimator of the residuals and of the normal density from DEM/USD (left) and GBP/USD (right). Quartic kernel, $\hat{h} = 2.78\hat{\sigma}n^{-0.2}$.

SFEresDens

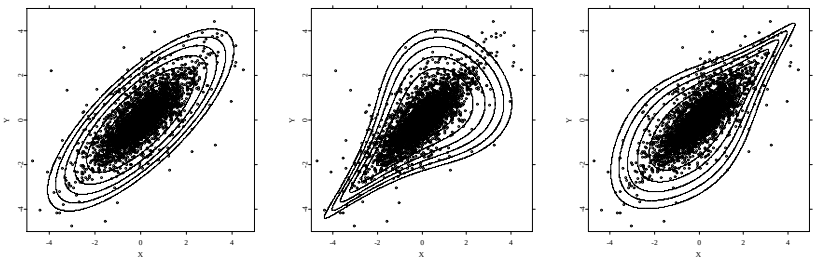


Figure 17.10: Residuals $\hat{\varepsilon}$ and fitted copulae: Gaussian ($\hat{\rho} = 0.76898$), Clayton ($\hat{\theta} = 1.8611$), Gumbel ($\hat{\theta} = 2.2833$).

SFEstaticCop

| | $\alpha(\times 10^2)$ | | | |
|---|---|---|---|---|
| Portfolio | 5 | 1 | 0.5 | 0.1 |
| $(1,1)$ | -0.030 | -0.042 | -0.046 | -0.055 |
| | -0.026 | -0.038 | -0.042 | -0.049 |
| | -0.026 | -0.043 | -0.051 | -0.078 |
| $(1,2)$ | -0.031 | -0.044 | -0.049 | -0.058 |
| | -0.029 | -0.043 | -0.048 | -0.056 |
| | -0.028 | -0.048 | -0.056 | -0.089 |
| $(1,3)$ | -0.033 | -0.046 | -0.052 | -0.062 |
| | -0.033 | -0.048 | -0.054 | -0.064 |
| | -0.031 | -0.053 | -0.062 | -0.099 |
| $(2,1)$ | -0.058 | -0.083 | -0.091 | -0.109 |
| | -0.049 | -0.071 | -0.079 | -0.093 |
| | -0.049 | -0.082 | -0.097 | -0.147 |
| $(2,3)$ | -0.061 | -0.086 | -0.095 | -0.113 |
| | -0.056 | -0.081 | -0.091 | -0.106 |
| | -0.054 | -0.090 | -0.108 | -0.168 |
| $(3,2)$ | -0.061 | -0.086 | -0.095 | -0.113 |
| | -0.075 | -0.109 | -0.122 | -0.143 |
| | -0.074 | -0.125 | -0.149 | -0.226 |
| $(-1,1)$ | -0.027 | -0.039 | -0.043 | -0.052 |
| | -0.026 | -0.031 | -0.034 | -0.041 |
| | -0.020 | -0.028 | -0.031 | -0.037 |
| $(-1,2)$ | -0.026 | -0.037 | -0.040 | -0.050 |
| | -0.020 | -0.029 | -0.034 | -0.040 |
| | -0.017 | -0.024 | -0.026 | -0.030 |
| $(-1,3)$ | -0.025 | -0.035 | -0.039 | -0.048 |
| | -0.019 | -0.029 | -0.032 | -0.040 |
| | -0.015 | -0.021 | -0.023 | -0.025 |
| $(-2,1)$ | -0.056 | -0.080 | -0.088 | -0.106 |
| | -0.044 | -0.064 | -0.070 | -0.084 |
| | -0.043 | -0.062 | -0.069 | -0.082 |
| $(-2,3)$ | -0.054 | -0.075 | -0.083 | -0.102 |
| | -0.042 | -0.061 | -0.068 | -0.081 |
| | -0.037 | -0.052 | -0.058 | -0.069 |
| $(-3,2)$ | -0.084 | -0.118 | -0.132 | -0.159 |
| | -0.066 | -0.096 | -0.105 | -0.125 |
| | -0.063 | -0.090 | -0.100 | -0.119 |

Table 17.4: $\widehat{VaR}(\alpha)$ for different portfolios and $\alpha$ values (static copulae). For each portfolio estimated with Gaussian, (first row), Clayton (second row) and Gumbel copula (third row).

| | $\alpha(\times 10^2)$ | | | |
|---|---|---|---|---|
| Portfolio | 5 | 1 | 0.5 | 0.1 |
| $(1,1)$ | 4.81 | 1.58 | 1.00 | 0.37 |
| $(1,2)$ | 4.61 | 1.41 | 0.92 | 0.34 |
| $(1,3)$ | 4.75 | 1.41 | 0.95 | 0.37 |
| $(2,1)$ | 5.07 | 1.81 | 1.03 | 0.43 |
| $(2,3)$ | 4.61 | 1.44 | 0.92 | 0.34 |
| $(3,2)$ | 4.98 | 1.64 | 1.03 | 0.43 |
| $(-1,1)$ | 3.51 | 0.72 | 0.34 | 0.14 |
| $(-1,2)$ | 1.84 | 0.37 | 0.23 | 0.11 |
| $(-1,3)$ | 1.96 | 0.46 | 0.23 | 0.11 |
| $(-2,1)$ | 4.18 | 1.06 | 0.72 | 0.20 |
| $(-2,3)$ | 2.76 | 0.43 | 0.17 | 0.14 |
| $(-3,2)$ | 3.83 | 0.89 | 0.57 | 0.17 |
| avg | 3.91 | 1.10 | 0.68 | 0.27 |
| std.dev. | 1.15 | 0.52 | 0.35 | 0.12 |

Table 17.5: Clayton copula, exceedances ratio $\hat{\alpha}(\times 10^2)$ for different portfolios.

ing results for each copula, portfolio and quantiles at levels $\alpha$ for $\alpha_1 = 0.05$, $\alpha_2 = 0.01$, $\alpha_3 = 0.005$ and $\alpha_4 = 0.001$ are in Tables 17.5, 17.6 and 17.7. One can observe, that the Gumbel copula produces better Backtesting results at levels $\alpha_1 = 0.05$ and $\alpha_2 = 0.01$, whereas Clayton copula performs better for smaller quantiles at levels $\alpha_3 = 0.005$ and $\alpha_4 = 0.001$. In addition, we can verify that on average the Clayton and the Gaussian copula overestimate overestimate VaR.

### 17.4.2 5-dimensional Exchange Rate Portfolio

In this section, the Value-at-Risk of exchange rate portfolios composed of 5 positions (USD value of GBP, FRF, CHF, DEM and AUD from 04.01.1994 to 15.08.1997) is computed using a time-varying Clayton copula.

The fit of a GARCH(1,1) model to the sample of log returns $\{x_t\}_{t=1}^T$, $x_t = (x_{1,t}, \ldots, x_{5,t})^\top$, $T = 907$, gives the estimates $\hat{\omega}_j$, $\hat{\alpha}_j$ and $\hat{\beta}_j$, as in Table 17.8, and empirical residuals $\{\hat{\varepsilon}_t\}_{t=1}^T$, where $\hat{\varepsilon}_t = (\hat{\varepsilon}_{1,t}, \ldots, \hat{\varepsilon}_{5,t})^\top$, as in upper right part of Figure 17.15. The marginal distributions are specified as normal, $\hat{\varepsilon}_j \sim N(\hat{\mu}_j, \hat{\sigma}_j)$, the estimated parameters $\hat{\delta}_j = (\hat{\mu}_j, \hat{\sigma}_j)$ are in Table 17.9.

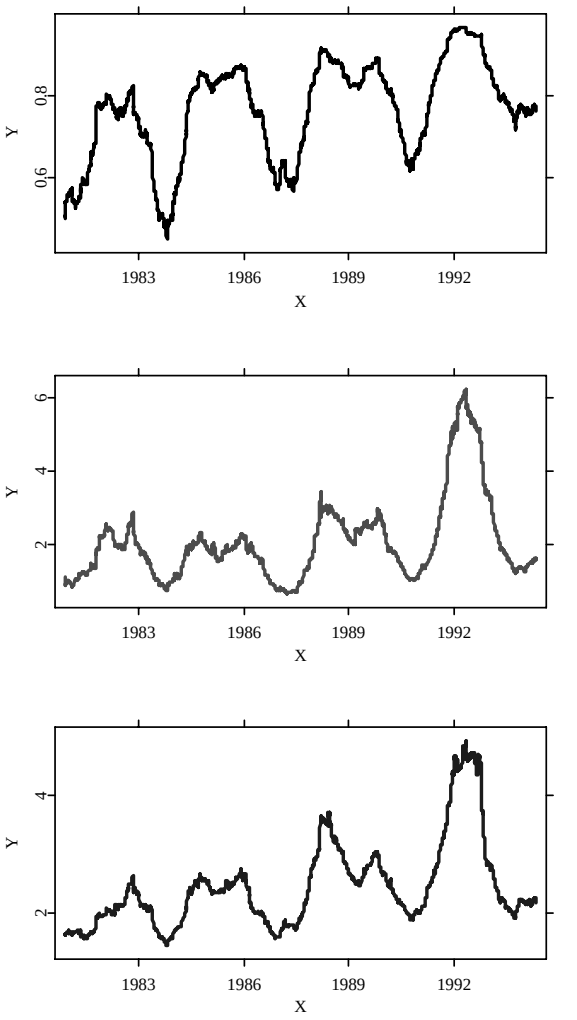


Figure 17.11: Dependence parameter $\hat{\theta}$, estimated using IFM method, Gaussian (upper panel), Clayton (middle panel) and Gumbel (lower panel) copulae, moving window ($w = 250$).

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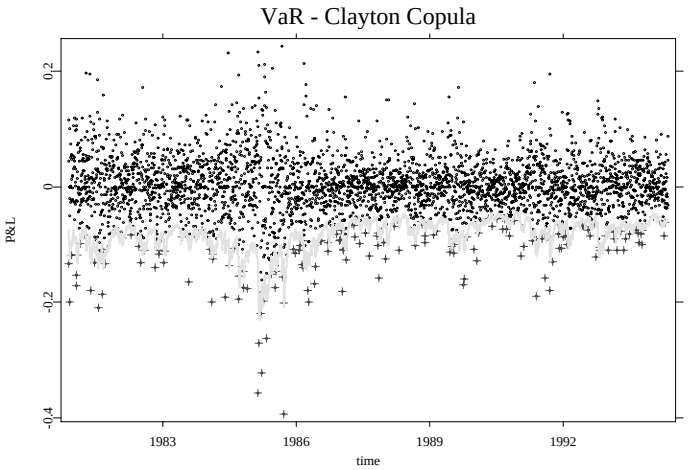


Figure 17.12: $\widehat{VaR}(\alpha)$ (solid line), P&L (dots) and exceedances (crosses), $\alpha = 0.05$, $\hat{\alpha} = 0.04987$, $w = (3, 2)^\top$. P&L samples generated with Clayton copula.

SFEclaytonSIM2ptv

| | $\alpha(\times 10^2)$ | | | |
|---|---|---|---|---|
| Portfolio | 5 | 1 | 0.5 | 0.1 |
| $(1, 1)$ | 5.21 | 1.09 | 0.43 | 0.09 |
| $(1, 2)$ | 5.16 | 1.03 | 0.43 | 0.09 |
| $(1, 3)$ | 4.92 | 0.98 | 0.49 | 0.09 |
| $(2, 1)$ | 5.21 | 1.03 | 0.49 | 0.12 |
| $(2, 3)$ | 5.16 | 1.00 | 0.49 | 0.09 |
| $(3, 2)$ | 5.21 | 1.06 | 0.46 | 0.12 |
| $(-1, 1)$ | 5.21 | 1.90 | 1.33 | 0.58 |
| $(-1, 2)$ | 5.96 | 1.67 | 1.04 | 0.46 |
| $(-1, 3)$ | 4.64 | 1.09 | 0.52 | 0.26 |
| $(-2, 1)$ | 5.10 | 1.67 | 1.12 | 0.52 |
| $(-2, 3)$ | 5.53 | 2.07 | 1.30 | 0.55 |
| $(-3, 2)$ | 5.01 | 1.72 | 1.15 | 0.52 |
| avg. | 5.20 | 1.36 | 0.77 | 0.29 |
| std.dev. | 0.32 | 0.41 | 0.38 | 0.22 |

Table 17.6: Gumbel copula, exceedances ratio $\hat{\alpha}(\times 10^2)$ for different portfolios.

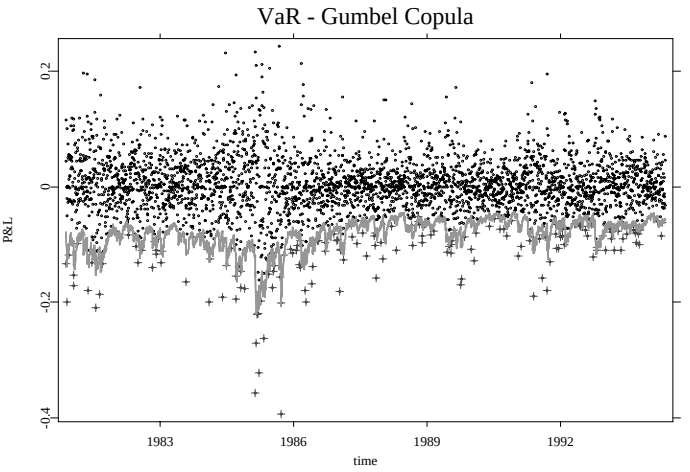


Figure 17.13: $\widehat{VaR}(\alpha)$ (solid line), P&L (dots) and exceedances (crosses), $\alpha = 0.05$, $\hat{\alpha} = 0.0521$, $w = (3,2)^\top$. P&L samples generated with Gumbel copula.

SFEgumbelSIM2ptv

| | $\alpha(\times 10^2)$ | | | |
|---|---|---|---|---|
| Portfolio | 5 | 1 | 0.5 | 0.1 |
| $(1,1)$ | 3.72 | 1.09 | 0.66 | 0.23 |
| $(1,2)$ | 5.13 | 1.64 | 1.21 | 0.52 |
| $(1,3)$ | 6.14 | 1.96 | 1.55 | 0.75 |
| $(2,1)$ | 3.29 | 0.78 | 0.58 | 0.14 |
| $(2,3)$ | 4.32 | 1.47 | 0.92 | 0.43 |
| $(3,2)$ | 3.34 | 0.86 | 0.63 | 0.20 |
| $(-1,1)$ | 1.28 | 0.23 | 0.14 | 0.09 |
| $(-1,2)$ | 0.84 | 0.17 | 0.12 | 0.01 |
| $(-1,3)$ | 1.04 | 0.32 | 0.20 | 0.01 |
| $(-2,1)$ | 1.99 | 0.35 | 0.17 | 0.09 |
| $(-2,3)$ | 0.98 | 0.23 | 0.14 | 0.09 |
| $(-3,2)$ | 1.76 | 0.32 | 0.14 | 0.09 |
| avg. | 2.81 | 0.80 | 0.54 | 0.23 |
| std.dev. | 1.75 | 0.63 | 0.48 | 0.21 |

Table 17.7: Gaussian copula, exceedances ratio $\hat{\alpha}(\times 10^2)$ for different portfolios.

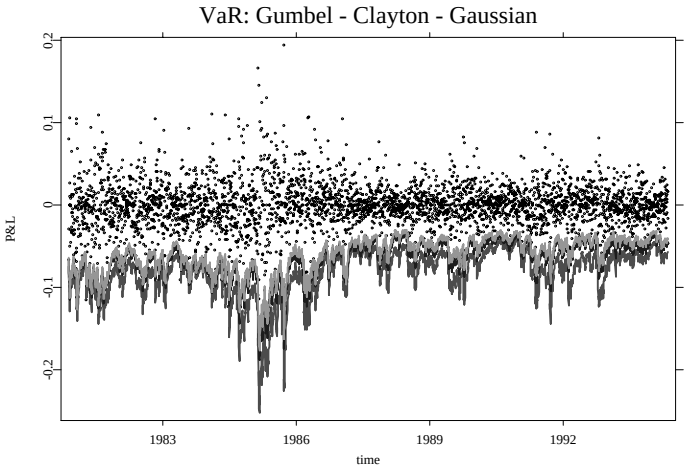


Figure 17.14: $\widehat{VaR}(\alpha)$ and P&L (dots), $\alpha = 0.01$, estimated with Gumbel copula, $\hat{\alpha} = 0.0167$, Clayton copula, $\hat{\alpha} = 0.0106$, and Gaussian copula, $\hat{\alpha} = 0.0034$, $w = (-2, 1)^\top$.

SFEClayGumbGauss

The estimated Value-at-Risk at level $\alpha$ together with the P&L function are plotted in Figure 17.16. Backtesting results for each portfolio for $\alpha_1 = 0.05$, $\alpha_2 = 0.01$, $\alpha_3 = 0.005$ and $\alpha_4 = 0.001$ are in Table 17.10.

To conclude, a summary of the main findings of this chapter. Three different copulae - Gumbel, Clayton and Gaussian - were used to estimate the Value-at-Risk from the 2-dimensional portfolio (DEM/USD, GBP/USD). From the time series of estimated dependence parameters, we can verify that the dependence structure is represented in a similar form with all copula families, as in Figure 17.11.

Using backtesting results to compare the performance in the VaR estimation, we remark that on average the Clayton and Gaussian copulae *overestimated* the VaR. In terms of capital requirement, a financial institution computing VaR with those copulae would be requested to keep *more* capital aside than necessary to guarantee the desired confidence level.

The estimation with Gumbel copula, on another side, produced results close to the desired level. Gumbel copulae seems to represent specific data dependence structures (like lower tail dependencies, relevant to explain simultane-

| | $\hat{\omega}$ | $\hat{\alpha}$ | $\hat{\beta}$ |
|---|---|---|---|
| $x_1$ | 0.000 | 0.043 | 0.931 |
| $x_2$ | 0.000 | 0.047 | 0.932 |
| $x_3$ | 0.000 | 0.099 | 0.813 |
| $x_4$ | 0.000 | 0.043 | 0.940 |
| $x_5$ | 0.000 | 0.002 | 0.000 |

Table 17.8: GARCH(1,1) parameters, 5-dimensional portfolio.

| | $\hat{\mu}(\times 10^2)$ | $\hat{\sigma}$ |
|---|---|---|
| $j = 1$ | 2.52 | 1.00 |
| $j = 2$ | -0.46 | 0.99 |
| $j = 3$ | -0.36 | 1.00 |
| $j = 4$ | -0.86 | 1.00 |
| $j = 5$ | 2.28 | 1.00 |

Table 17.9: Parameters from marginal distributions.

| | $\alpha(\times 10^2)$ | | | |
|---|---|---|---|---|
| Portfolio | 5 | 1 | 0.5 | 0.1 |
| $(1,1,1,1,1)$ | 5.02 | 0.61 | 0.47 | 0.15 |
| $(1,2,3,2,1)$ | 5.78 | 0.91 | 0.47 | 0.47 |
| $(1,3,1,2,3)$ | 3.96 | 0.47 | 0.47 | 0.30 |
| $(2,1,2,3,1)$ | 5.33 | 0.91 | 0.61 | 0.47 |
| $(2,1,3,2,1)$ | 5.63 | 0.91 | 0.47 | 0.47 |
| $(2,3,1,1,2)$ | 3.96 | 0.76 | 0.61 | 0.15 |
| $(2,3,3,2,1)$ | 5.78 | 0.91 | 0.47 | 0.47 |
| $(3,1,2,1,3)$ | 3.96 | 0.76 | 0.61 | 0.15 |
| $(3,1,2,2,2)$ | 4.87 | 0.76 | 0.61 | 0.15 |
| $(3,2,3,2,3)$ | 4.57 | 0.61 | 0.61 | 0.15 |
| avg. | 4.79 | 0.75 | 0.50 | 0.38 |
| std.dev. | 0.77 | 0.15 | 0.07 | 0.08 |

Table 17.10: Clayton copula, exceedances ratio $\hat{\alpha}$ for different portfolios.

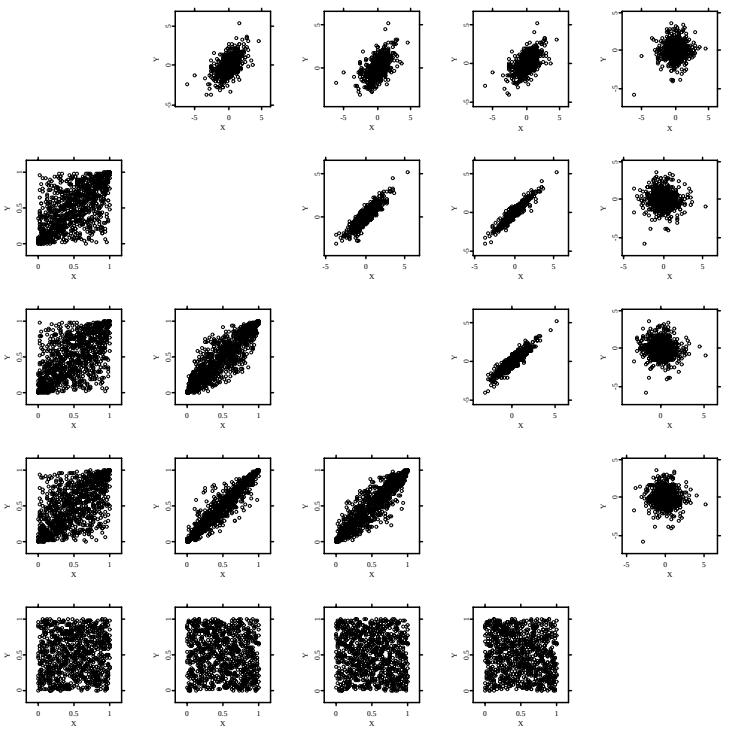

Figure 17.15: Scatterplots from GARCH residulas (upper triangular) and from residuals mapped on unit square by the cdf (lower triangular).

SFE5dim

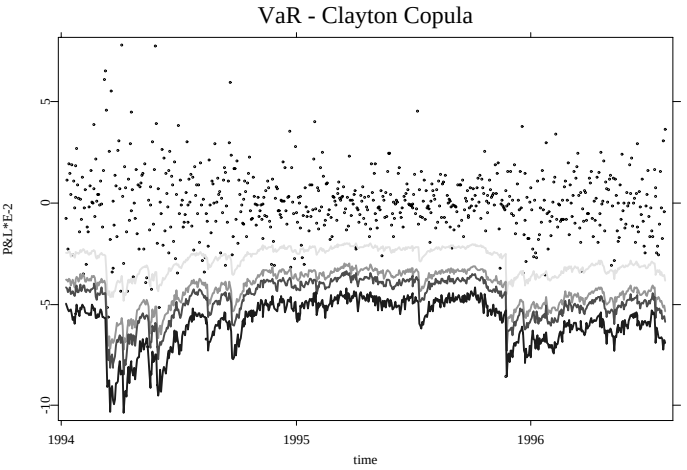


Figure 17.16: $\widehat{VaR}(\alpha)$ and P&L (dots), estimated with Clayton copula, $\alpha_1 = 0.05$, $\alpha_2 = 0.01$, $\alpha_3 = 0.005$ and $\alpha_4 = 0.001$, $w = (3, 1, 2, 1, 3)^\top$.
SFE5dim

ous losses) better than Gaussian and Clayton copulae.

## 17.5 Recommended Literature

The term copula goes back to the works of Sklar (1959) and Sklar (1996). A detailed discussion with proofs and deep mathematical treatment can be found in Joe (1997) and Nelsen (1999).

Nelsen (1999) features a theoretical introduction to copulae. Further, Nelsen (1999) generalises the concepts of Archimedean copulae for the $d$-dimensional case. For more copula families and respective properties, refer to Joe (1997).

A practical introduction is given in Deutsch and Eller (1999). Embrechts, McNeil and Straumann (1999b) discuss restrictions of the copula technique and their relation to the classical correlation analysis.

Different techniques for the simulation from $d$ pseudo random variables with joint distribution defined by a copula $C$ and $d$ marginal distributions are used. A standard method for Archimedean copulae (the conditional distribution method) as well as other different methods are described in Bouyé,

Durrleman, Nikeghbali, Riboulet and Roncalli (2000), Devroye (1986) and Embrechts et al. (2005).

Detailed description of the VaR estimation procedure at prescribed level $\alpha$ is to be found in Giacomini and Härdle (2005).

The theory for copula model selection tests is developed in Chen and Fan (2006), Chen and Fan (2004) and Chen, Fan and Tsyrennikov (2004) for static set up.

# 18 Statistics of Extreme Risks

When we model returns using a GARCH process with normally distributed innovations, we have already taken into account the second *stylised fact* (see Chapter 13). The distribution of the random returns automatically has a leptokurtosis and larger losses occurring more frequently than under the assumption that the returns are normally distributed. If one is interested in the 95%-VaR of liquid assets, this approach produces the most useful results. For the extreme risk quantiles such as the 99%-VaR and for riskier types of investments the risk is often underestimated when the innovations are assumed to be normally distributed, since a higher probability of particularly extreme losses than a GARCH process $\varepsilon_t$ with normally distributed $Z_t$ can produce.

Thus procedures have been developed which assume that the tails of the innovation's distribution are heavier. The probability of extreme values largely depends on how slowly the probability density function $f_Z(x)$ of the innovations goes to 0 as $|x| \to \infty$. The rate at which it diminishes must be estimated from the data. Since extreme observations are rare, this produces a difficult estimation problem. Even large data sets contain only limited information on the true probability of an extreme loss (profit). In such a situation methods from extreme value statistics produce a more realistic estimate of the risk. In this chapter a short overview of the basic ideas and several of the latest applications are given.

## 18.1 Limit Behaviour of Maxima

Consider the stochastic behaviour of the maximum $M_n = \max(X_1, \ldots, X_n)$ of $n$ identically distributed random variables $X_1, \ldots, X_n$ with cumulative distribution function (cdf) $F(x)$. From a risk management perspective $X_t = -Z_t$ is the negative return at day $t$. The cdf of $M_n$ is

$$\mathrm{P}(M_n \leq x) = \mathrm{P}(X_1 \leq x, \ldots, X_n \leq x) = \prod_{t=1}^{n} \mathrm{P}(X_t \leq x) = F^n(x). \quad (18.1)$$

We are only considering unbounded random variables $X_t$, i.e. $F(x) < 1$ for all $x < \infty$. Obviously it holds that $F^n(x) \to 0$ for all $x$, when $n \to \infty$, and thus $M_n \xrightarrow{P} \infty$. The maximum of $n$ unbounded random variables increases over all boundaries. In order to achieve a non-degenerate behaviour limit, $M_n$ has to be standardised in a suitable fashion.

**Definition 18.1 (Maximum Domain of Attraction)**
*The random variable $X_t$ belongs to the* maximum domain of attraction *(MDA) of a non-degenerate distribution $G$, if for suitable sequences $c_n > 0, d_n$ it holds that:*

$$\frac{M_n - d_n}{c_n} \xrightarrow{\mathcal{L}} G \quad \text{for } n \to \infty,$$

*i.e. $F^n(c_n x + d_n) \to G(x)$ at all continuity points $x$ of the cdf $G(x)$.*

It turns out that only a few distributions $G$ can be considered as the asymptotic limit distribution of the standardised maximum $M_n$. They are referred to as the *extreme value distriubtions*. These are the following three distribution functions:

*Fréchet:* $G_{1,\alpha}(x) = \exp\{-x^{-\alpha}\},\ x \geq 0$ , for $\alpha > 0$,

*Gumbel:* $G_0(x) = \exp\{-e^{-x}\},\ x \in \mathbb{R}$,

*Weibull:* $G_{2,\alpha}(x) = \exp\{-|x|^{-\alpha}\},\ x \leq 0$ , for $\alpha < 0$.

The Fréchet distributions are concentrated on the non-negative real numbers $[0, \infty)$, while the Weibull distribution, on the other hand, on $(-\infty, 0]$, whereas the Gumbel distributed random variables can attain any real number. Figure 18.1 displays the density function of the Gumbel distribution, the Fréchet distribution with parameter $\alpha = 2$ and the Weibull distribution with parameter $\alpha = -2$. All three distributions types can be displayed in a single Mises form:

**Definition 18.2 (Extreme Value Distributions)**
*The* generalised extreme value distribution *(GEV =* generalised extreme value*) with the* form parameter $\gamma \in \mathbb{R}$ *has the distribution function:*

$$G_\gamma(x) = \exp\{-(1+\gamma x)^{-1/\gamma}\},\ 1 + \gamma x > 0 \ \text{for } \gamma \neq 0$$
$$G_0(x) = \exp\{-e^{-x}\},\ x \in \mathbb{R}$$

$G_0$ is the Gumbel distribution, whereas $G_\gamma, \gamma \neq 0$ is linked to the Fréchet- and Weibull distributions by the following relationships:

$$G_\gamma(\frac{x-1}{\gamma}) = G_{1,1/\gamma}(x) \text{ for } \gamma > 0,$$

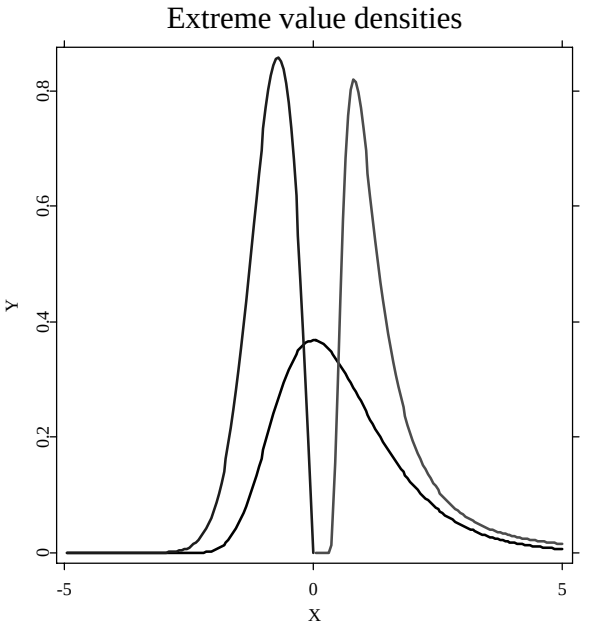


Figure 18.1: Fréchet (red), Gumbel (black) and Weibull distributions (blue). SFEevt1

$$G_\gamma(-\frac{x+1}{\gamma}) = G_{2,-1/\gamma}(x) \text{ for } \gamma < 0.$$

This definition describes the standard form of the GEV distributions. In general we can change the centre and the scale to obtain other GEV distributions: $G(x) = G_\gamma(\frac{x-\mu}{\sigma})$ with the form parameter $\gamma$, the location parameter $\mu \in \mathbb{R}$ and the scale parameter $\sigma > 0$. For asymptotic theory this does not matter since the standardised sequences $c_n, d_n$ can be always chosen so that the asymptotic distribution $G$ has the standard form ($\mu = 0, \sigma = 1$). An important result of the asymptotic distribution of the maximum $M_n$ is the *Fisher-Tippett theorem*:

**Theorem 18.1**
*If sequences $c_n > 0, d_n$ exist and a non-degenerate distribution $G$, so that*

$$\frac{M_n - d_n}{c_n} \xrightarrow{\mathcal{L}} G \quad \text{for } n \to \infty,$$

*then $G$ is a GEV distribution.*

**Proof:**
As a form of clarification the basic ideas used to prove this central result are outlined. Let $t > 0$, and $[z]$ represent the integer part of $z$. Since $F^{[nt]}$ is the distribution function of $M_{[nt]}$, due to our assumptions on the asymptotic distribution of $M_n$ it holds that

$$F^{[nt]}(c_{[nt]}x + d_{[nt]}) \longrightarrow G(x) \text{ for } [nt] \to \infty, \text{ i.e. } n \to \infty.$$

On the other hand it also holds that

$$F^{[nt]}(c_n x + d_n) = \{F^n(c_n x + d_n)\}^{\frac{[nt]}{n}} \longrightarrow G^t(x) \text{ for } n \to \infty.$$

In other words this means that

$$\frac{M_{[nt]} - d_{[nt]}}{c_{[nt]}} \xrightarrow{\mathcal{L}} G\,, \qquad \frac{M_{[nt]} - d_n}{c_n} \xrightarrow{\mathcal{L}} G^t$$

for $n \to \infty$. According to the Lemma, which is stated below, this is only possible when

$$\frac{c_n}{c_{[nt]}} \longrightarrow b(t) \geq 0, \qquad \frac{d_n - d_{[nt]}}{c_{[nt]}} \longrightarrow a(t)$$

and

$$G^t(x) = G(b(t)x + a(t)),\ t > 0,\ x \in \mathbb{R}. \tag{18.2}$$

This relationship holds for arbitrary values $t$. We use it in particular for arbitrary $t, s$ and $s \cdot t$ and obtain

$$b(st) = b(s)\ b(t),\ a(st) = b(t)a(s) + a(t). \tag{18.3}$$

The functional equations (18.2), (18.3) for $G(x), b(t), a(t)$ have only one solution, when $G$ is one of the distributions $G_0, G_{1,\alpha}$ or $G_{2,\alpha}$, that is, $G$ must be a GEV distribution.

□

**Lemma 18.1 (Convergence Type Theorem)**
*Let $U_1, U_2, \ldots, V, W$ be random variables, $b_n, \beta_n > 0$, $a_n, \alpha_n \in \mathbb{R}$. If*

$$\frac{U_n - a_n}{b_n} \xrightarrow{\mathcal{L}} V$$

*in distribution for $n \to \infty$, then it holds that:*

$$\frac{U_n - \alpha_n}{\beta_n} \xrightarrow{\mathcal{L}} W \qquad \textit{if and only if} \qquad \frac{b_n}{\beta_n} \longrightarrow b \geq 0, \ \frac{a_n - \alpha_n}{\beta_n} \longrightarrow a \in \mathbb{R}.$$

*In this case $W$ has the same distribution as $bV + a$.*

Notice that the GEV distributions are identical to the so called *max-stable* distributions, by which for all $n \geq 1$ the maximum $M_n$ of $n$ i.i.d. random variables $X_1, \ldots, X_n$ have the same distribution as $c_n X_1 + d_n$ for appropriately chosen $c_n > 0, d_n$.

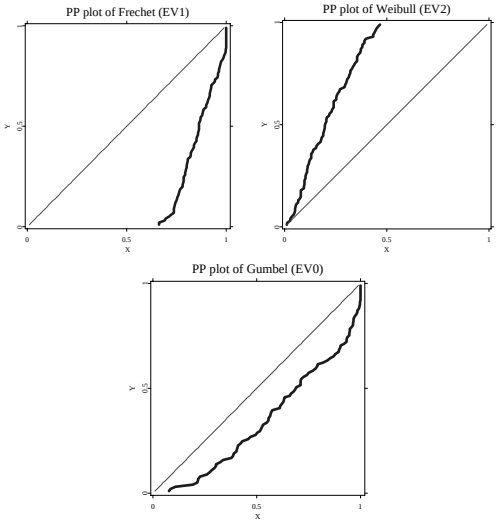


Figure 18.2: PP plot for the normal distribution and pseudo random variables with extreme value distributions. Fr'echet (upper left), Weibull (upper right) and Gumbel (below). SFEevt2

Figure 18.2 shows the so called normal plot, i.e., it compares the graph of the cdf of the normal distribution with the one in Section 18.2 for the special case $F(x) = \Phi(x)$ with computer generated random variables that have a Gumbel distribution, Fréchet distribution with parameter $\alpha = 2$ and a Weibull distribution with parameter $\alpha = -2$ respectively. The differences with the normally distributed random variables, which would have approximately a straight line in a normal plot, can be clearly seen.

If the maximum of i.i.d. random variables converges in distribution after being appropriately standardised, then the question arises which of the three GEV distributions is the asymptotic distribution. The deciding factor is how fast the probability for extremely large observations decreases beyond a threshold $x$, when $x$ increases. Since this exceedance probability plays an important role in extreme value theory, we will introduce some more notations:

$$\overline{F}(x) = \mathrm{P}(X_1 > x) = 1 - F(x).$$

The relationship between the exceedance probability $\overline{F}(x)$ and the distribution of the maxima $M_n$ will become clear with the following theorem.

**Theorem 18.2**
*a) For $0 \le \tau \le \infty$ and every sequence of real numbers $u_n, n \ge 1$, it holds for $n \to \infty$ that*

$$n\overline{F}(u_n) \to \tau \quad \textit{if and only if} \quad \mathrm{P}(M_n \le u_n) \to e^{-\tau}.$$

*b) $F$ belongs to the maximum domain of attraction of the GEV distribution $G$ with the standardised sequences $c_n, d_n$ exactly when $n \to \infty$*

$$n\overline{F}(c_n x + d_n) \to -\log G(x) \quad \textit{for all } x \in \mathbb{R}.$$

The exceedance probability of the Fréchet distribution $G_{1,\alpha}$ behaves like $1/x^\alpha$ for $x \to \infty$, because the exponential function around 0 is approximately linear, i.e.,

$$\overline{G}_{1,\alpha}(x) = \frac{1}{x^\alpha}\{1 + o(1)\} \qquad \text{for} \quad x \to \infty.$$

Essentially all of the distributions that belong to the MDA of this Fréchet distribution show the same behaviour; $x^\alpha \overline{F}(x)$ is almost constant for $x \to \infty$, or more specifically: a slowly varying function.

**Definition 18.3**
*A positive measurable function $L$ in $(0, \infty)$ is called* slowly varying*, if for all $t > 0$*

$$\frac{L(tx)}{L(x)} \to 1 \qquad \textit{for} \quad x \to \infty.$$

Typical slowly varying functions are, in addition to constants, logarithmic growth rates, for example $L(x) = \log(1 + x), x > 0$.

**Theorem 18.3**
*$F$ belongs to the maximum domain of attraction of the Fréchet distribution*

*$G_{1,\alpha}$ for some $\alpha > 0$, if and only if $x^\alpha \overline{F}(x) = L(x)$ is a slowly varying function. The random variables $X_t$ with the distribution function $F$ are unbounded, i.e., $F(x) < 1$ for all $x < \infty$, and it holds that*

$$\frac{M_n}{c_n} \xrightarrow{\mathcal{L}} G_{1,\alpha}$$

*with $c_n = F^{-1}(1 - \frac{1}{n})$.*

For the description of the standardised sequence $c_n$ we have used the following notation. $c_n$ is an extreme quantile of the distribution $F$, and it holds that $\overline{F}(c_n) = \mathrm{P}(X_t > c_n) = 1/n$.

**Definition 18.4 (Quantile Function)**
*If $F$ is a distribution function, we call the generalised inverse*

$$F^{-1}(\gamma) = \inf\{x \in \mathbb{R};\ F(x) \geq \gamma\},\ \ 0 < \gamma < 1,$$

*the* quantile function. *It then holds that $\mathrm{P}(X_1 \leq F^{-1}(\gamma)) = \gamma$, i.e., $F^{-1}(\gamma)$ is the $\gamma$-*quantile *of the distribution $F$.*

If $F$ is strictly monotonic increasing and continuous, then $F^{-1}$ is the generalised inverse of $F$.

There is a corresponding criterion for the Weibull distribution that can be shown using the relationship $G_{2,\alpha}(-x^{-1}) = G_{1,\alpha}(x),\ x > 0$,. Random variables, whose maxima are asymptotically Weibull distributed, are by all means bounded, i.e., a constant $c < \infty$ exists, such that $X_t \leq c$ with probability 1. Therefore, in financial applications they are only interesting in special situations where using a type of hedging strategy, the loss, which can result from an investment, is limited. In order to prohibit continuous differentiations in various cases, below we will mainly discuss case where the losses are unlimited. Cases in which losses are limited can be dealt with in a similar fashion.

Fréchet distributions appear as asymptotic distributions of the maxima of those random variables whose probability of values beyond $x$ only slowly decreases with $x$, whereas only bounded random variables belong to the maximum domain of attraction of Weibull distributions. Many known distributions such as the exponential or the normal distribution do not belong to either one of the groups. It is likely that in such cases the distribution of the appropriate standardised maxima converges to a Gumbel distribution. The general conditions needed for this are however more complicated and more difficult to prove than they were for the Fréchet distribution.

**Theorem 18.4**
*The distribution function $F$ of the unbounded random variables $X_t$ belongs to the maximum domain of attraction of the Gumbel distribution if measurable scaling functions $c(x), g(x) > 0$ as well as an absolute continuous function $e(x) > 0$ exist with $c(x) \to c > 0,\ g(x) \to 1, e'(x) \to 0$ for $x \to \infty$ so that for $z < \infty$*

$$\overline{F}(x) = c(x) \exp\{-\int_z^x \frac{g(y)}{e(y)} dy\},\ z < x < \infty.$$

*In this case it holds that*

$$\frac{M_n - d_n}{c_n} \xrightarrow{\mathcal{L}} G_0$$

*with $d_n = F^{-1}(1 - \frac{1}{n})$ and $c_n = e(d_n)$.*

As a function $e(x)$, the *average excess function* can be used:

$$e(x) = \frac{1}{\overline{F}(x)} \int_x^\infty \overline{F}(y)\, dy,\ \ x < \infty,$$

which will be considered in more detail in the following.

The exponential distribution with parameter $\lambda$ has the distribution function $F(x) = 1 - e^{-\lambda x}, x \geq 0$, so that $\overline{F}(x) = e^{-\lambda x}$ fulfills the conditions stipulated in the theorem with $c(x) = 1,\ g(x) = 1,\ z = 0$ and $e(x) = 1/\lambda$. The maximum $M_n$ of $n$ independent exponentially distributed random variables with parameter $\lambda$ thus converges in distribution to the Gumbel distribution:

$$\lambda(M_n - \frac{1}{\lambda}\ \log\ n) \xrightarrow{\mathcal{L}} G_0 \quad \text{for} \quad n \to \infty.$$

In general, however, the conditions are not so easy to check. There are other simple sufficient conditions with which it can be shown, for example, that the normal distribution also belongs to the maximum domain of attraction of the Gumbel distribution. If, for example, $M_n$ is the maximum of $n$ independent standard normally distributed random variables, then it holds that

$$\sqrt{2\ \log\ n}(M_n - d_n) \xrightarrow{\mathcal{L}} G_0 \quad \text{for} \quad n \to \infty$$

$$\text{with} \qquad d_n = \sqrt{2\ \log\ n} - \frac{\log\log\ n + \log(4\pi)}{2\sqrt{2\ \log\ n}}.$$

SFEevtex1

Another member of the distributions in the maximum domain of attraction of the Fréchet distribution $G_{1,\alpha}$ is the *Pareto distribution* with the distribution function

$$W_{1,\alpha}(x) = 1 - \frac{1}{x^\alpha}, x \geq 1, \alpha > 0,$$

as well as all other distributions with *Pareto tails*, i.e., with

$$\overline{F}(x) = \frac{\kappa}{x^\alpha}\{1 + o(1)\} \quad \text{for} \quad x \to \infty.$$

Since $\overline{F}^{-1}(\gamma)$ for $\gamma \approx 1$ behaves here like $(\kappa/\gamma)^{1/\alpha}$, $c_n$ for $n \to \infty$ is identical to $(\kappa n)^{1/\alpha}$, and

$$\frac{M_n}{(\kappa n)^{1/\alpha}} \xrightarrow{\mathcal{L}} G_{1,\alpha} \quad \text{for } n \to \infty.$$

There is a tight relationship between the asymptotic behaviour of the maxima of random variables and the distribution of the corresponding excesses which builds the foundation for an important estimation method in the extreme value statistic, which is defined in the next section. In general it deals with observations crossing a specified threshold $u$. Their distribution $F_u$ is defined as follows:

**Definition 18.5 (Excess Distribution)**
*Let $u$ be an arbitrary threshold and $F$ a distribution function of an unbounded random variable $X$.*

*a)* $F_u(x) = \mathrm{P}\{X - u \le x \mid X > u\} = \{F(u+x) - F(u)\}/\overline{F}(u),\ 0 \le x < \infty$ *is called the* excess distribution *beyond the threshold $u$.*

*b)* $e(u) = \mathsf{E}\{X - u \mid X > u\},\ 0 < u < \infty$, *is the* average excess function.

With partial integration it follows that this definition of the average excess function together with the following Theorem 18.4 agrees with:

$$e(u) = \int_u^\infty \frac{\overline{F}(y)}{\overline{F}(u)} dy.$$

If $\Delta_u$ is a random variable with the distribution function $F_u$, then its expectation is $\mathsf{E}\Delta_u = e(u)$.

**Theorem 18.5**
*$X$ is a positive, unbounded random variable with an absolutely continuous distribution function $F$.*
*a) The average excess function $e(u)$ identifies $F$ exactly:*

$$\overline{F}(x) = \frac{e(0)}{e(x)} \exp\{-\int_0^x \frac{1}{e(u)} du\},\ \ x > 0.$$

*b) If $F$ is contained in the MDA of the Fréchet distribution $G_{1,\alpha}$, then $e(u)$ is approximately linear for $u \to \infty$: $e(u) = \frac{1}{\alpha - 1}\ u\{1 + o(1)\}$.*

**Definition 18.6 (Pareto Distribution)**
*The* generalised Pareto distribution *(GP =* generalised Pareto*) with parameters $\beta > 0$, $\gamma$ has the distribution function*

$$W_{\gamma,\beta}(x) = 1 - (1 + \frac{\gamma x}{\beta})^{-\frac{1}{\gamma}} \quad \textit{for} \quad \begin{cases} x \geq 0 & \textit{if} \quad \gamma > 0 \\ 0 \leq x \leq \frac{-\beta}{\gamma} & \textit{if} \quad \gamma < 0, \end{cases}$$

*and*

$$W_{0,\beta}(x) = 1 - e^{-\frac{1}{\beta}x}, \ x \geq 0.$$

$W_\gamma(x) = W_{\gamma,1}(x)$ *is called the* generalised standard Pareto distribution *or* standardised *GP distribution.*

Figure 18.3 shows the generalised standard Pareto distribution with parameters $\gamma = 0.5, 0$ and $-0.5$ respectively.

For $\gamma = 0$ the standardised GP distribution is an exponential distribution with parameter 1. For $\gamma > 0$ it is a Pareto distribution $W_{1,\alpha}$ with the parameter $\alpha = 1/\gamma$. For $\gamma < 0$ the GP distribution is also referred to as a *Beta distribution* and has the distribution function $W_{2,\alpha} = 1 - (-x)^{-\alpha}, -1 \leq x \leq 0, \alpha < 0$.

**Theorem 18.6**
*The distribution $F$ is contained in the MDA of the GEV distribution $G_\gamma$ with the form parameter $\gamma \geq 0$, exactly when for a measurable function $\beta(u) > 0$ and the GP distribution $W_{\gamma,\beta}$ it holds that:*

$$\sup_{x \geq 0} |F_u(x) - W_{\gamma,\beta(u)}(x)| \to 0 \, \textit{for} \ u \to \infty.$$

A corresponding result also holds for the case when $\gamma < 0$, in which case the supremum of $x$ must be taken for those $0 < W_{\gamma,\beta(u)}(x) < 1$.

For the generalised Pareto distribution $F = W_{\gamma,\beta}$ it holds for every finite threshold $u > 0$

$$F_u(x) = W_{\gamma,\beta+\gamma u}(x) \quad \text{for} \quad \begin{cases} x \geq 0 & \text{if} \quad \gamma \geq 0 \\ 0 \leq x < -\frac{\beta}{\gamma} - u & \text{if} \quad \gamma < 0, \end{cases}$$

In this case $\beta(u) = \beta + \gamma\, u$.

## 18.2 Statistics of Extreme Events

Throughout the entire section $X, X_1, \ldots, X_n$ are unbounded, i.i.d. random variables with distribution function $F$.

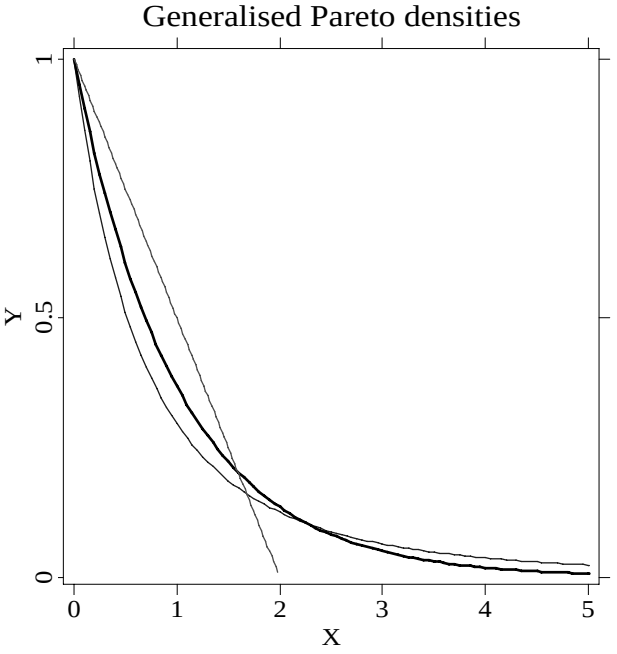


Figure 18.3: Standard pareto distribution ($\beta = 1$) with parameter $\gamma = 0.5$ (red), 0 (black) and $-0.5$ (blue). SFEgpdist

**Notation:** $X_{(1)} \leq \ldots \leq X_{(n)}$ and $X^{(1)} \geq \ldots \geq X^{(n)}$ represent the *order statistics*, that is, the data is sorted according to increasing or decreasing size. Obviously then $X_{(1)} = X^{(n)},\ X_{(n)} = X^{(1)}$ etc.

**Definition 18.7 (Empirical Average Excess Function)**
*Let $K_n(u) = \{j \leq n;\ X_j > u\}$ be the index of the observations outside of the threshold $u$, and let $N(u) = \#K_n(u)$ be their total number and*

$$\hat{F}_n(x) = \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}(X_j \leq x)$$

*the* empirical distribution function, $\overline{\hat{F}_n} = 1 - \hat{F}_n$.

$$e_n(u) = \int_u^\infty \overline{\hat{F}_n}(y)dy / \overline{\hat{F}_n}(u) = \frac{1}{N(u)} \sum_{j \in K_n(u)} (X_j - u)$$
$$= \frac{1}{N(u)} \sum_{j=1}^{n} max\{(X_j - u), 0\}$$

*is called the* empirical average excess function.

$e_n(u)$ estimates the average excess function $e(u)$ from Section 18.1.

As an explorative data analysis the following graphs will be considered:

| | |
|---|---|
| Plot of the probability distribution function | $\left\{F(X^{(k)}),\ \frac{n-k+1}{n+1}\right\}_{k=1}^{n}$, |
| Quantile plot | $\left\{X^{(k)},\ F^{-1}(\frac{n-k+1}{n+1})\right\}_{k=1}^{n}$, |
| Average excess plot | $\left\{X^{(k)}, e_n(X^{(k)})\right\}_{k=1}^{n}$. |

If the original model assumptions, that $F$ is the distribution of the data, is correct, then the first two graphs should be approximately linear. If this is not the case, then the distribution assumptions must be changed. On the other hand, due to Theorem 18.5, b) the average excess plot for size $k$ is approximately linear with a slope $1/(\alpha - 1)$ if $F$ belongs to the maximum domain of attraction of a Fréchet distribution $G_{1,\alpha}$ for $\alpha > 1$, i.e. with a finite expectation.

As an example consider the daily returns of the exchange rate between the Yen and the U.S. dollar from 1 December, 1978 to 31 January, 1991 in Figure 18.4. Figure 18.5 shows the plot of the probability distribution function and the quantile plot for the pdf $F(x) = \Phi(x)$ of the standard normal. The deviations from the straight line clearly shows that the data is not normally distributed. Figure 18.6 again shows the average excess plot of the data.

### 18.2.1 The POT (Peaks-Over-Threshold) Method

In this section and in the following one we will take a look at estimators for extreme value characteristics such as the exceedance probabilities $\overline{F}(x) = 1 - F(x)$ for values $x$ or the extreme quantile $F^{-1}(q)$ for $q \approx 1$.

First, we only consider distributions $F$ that are contained in the MDA of a GEV distribution $G_\gamma,\ \gamma \geq 0$,. The corresponding random variables are thus unbounded.

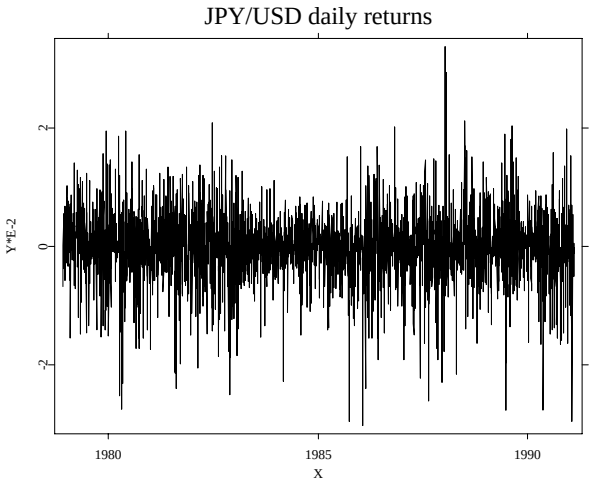


Figure 18.4: Daily log-return of JPY/USD exchange rate. SFEjpyusd

**Definition 18.8 (Excess)**
*Let $K_n(u)$ and $N(u)$ be, as before, the index and total number of observations beyond the threshold $u$ respectively. The* excess *beyond the threshold $u$ is defined as the random variables $Y_l, l = 1, \ldots, N(u)$, with*

$$\{Y_1, \ldots, Y_{N(u)}\} = \{X_j - u;\ j \in K_n(u)\} = \{X^{(1)} - u, \ldots, X^{(N(u))} - u\}.$$

The excesses $Y_l$, $l \leq \mathrm{N}(u)$ describe by how much the observations, which are larger than $u$, go beyond the threshold $u$. The POT method (*peaks-over-threshold method*) assumes that these excesses are the basic information source for the initial data. From the definition it immediately follows that $Y_1, \ldots, Y_{N(u)}$ are i.i.d. random variables with distribution $F_u$ given their random total number $\mathrm{N}(u)$, i.e., the excess distribution from Definition 18.5 is the actual distribution of the excesses. Due to Theorem 18.6 it also holds that $F_u(y) \approx W_{\gamma,\beta(u)}(y)$ for a GP distribution $W_{\gamma,\beta(u)}$ and all sufficiently large $u$.

Let's first consider the problem of estimating the exceedance probability $\overline{F}(x)$ for large $x$. A natural estimator is $\hat{\overline{F}}_n(x)$, the cdf at $x$ is replaced with the empirical distribution function. For large $x$, however, the empirical distribution function varies a lot because it is determined by the few extreme observations which are located around $x$. The effective size of the sub-sample of extreme, large observations is too small to use a pure non-parametric estimator such

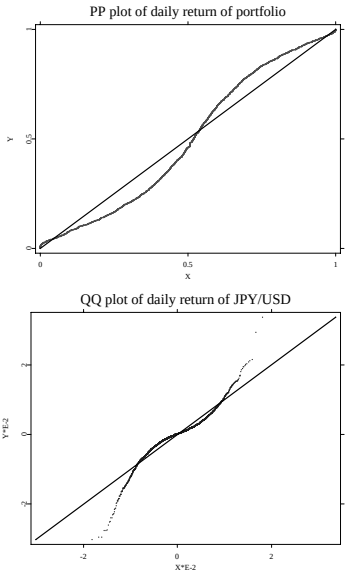


Figure 18.5: PP plot and QQ plot. SFEjpyusd

as the empirical distribution function. We therefore use the following relationship among the extreme exceedance probability $\overline{F}(x)$, the exceedance probability $\overline{F}(u)$ for a large, but not extremely large threshold and the excess distribution. Due to Definition 18.5 the excess distribution is

$$\begin{aligned} &\overline{F}_u(y) = \mathrm{P}(X - u > y \mid X > u) = \overline{F}(y+u)/\overline{F}(u), \qquad \text{i.e.} \\ &\overline{F}(x) = \overline{F}(u) \cdot \overline{F_u}(x-u), \;\; u < x < \infty. \end{aligned} \tag{18.4}$$

For large $u$ and using Theorem 18.6 we can approximate $F_u$ with $W_{\gamma,\beta}$ for appropriately chosen $\gamma, \beta$. $F(u)$ is replaced with the empirical distribution function $\hat{F}_n(u)$ at the threshold $u$, for which due to the definition of $\mathrm{N}(u)$ it holds that

$$\hat{F}_n(u) = \frac{n - \mathrm{N}(u)}{n} = 1 - \frac{\mathrm{N}(u)}{n}.$$

For $u$ itself this is a useful approximation, but not for the values $x$, which are clearly larger than the average sized threshold $u$. The estimator $1 - \hat{F}_n(x)$ of $\overline{F}(x)$ for extreme $x$ only depends on a few observations and is therefore too unreliable. For this reason the POT method uses the identity (18.4) for $\overline{F}(x)$ and replaces both factors on the right hand side with their corresponding

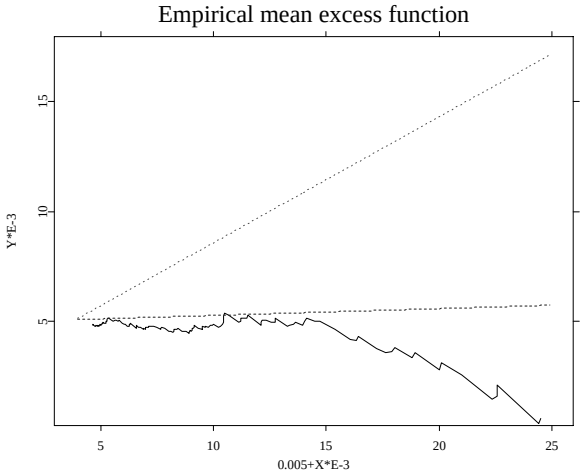


Figure 18.6: Empirical mean excess function (solid line), GP mean excess function for Hill estimator (dotted line) and moment estimator (broken line). SFEjpyusd

approximations, whereby the unknown parameter of the generalised Pareto distribution is replaced with a suitable estimator.

**Definition 18.9 (POT Estimator)**
*The* POT estimator $\overline{F}^{\wedge}(x)$ *for the exceedance probability* $\overline{F}(x)$, *for large* $x$, *is*

$$\overline{F}^{\wedge}(x) = \frac{N(u)}{n}\,\overline{W}_{\hat{\gamma},\hat{\beta}}(x-u) = \frac{N(u)}{n}\left\{1+\frac{\hat{\gamma}(x-u)}{\hat{\beta}}\right\}^{-1/\hat{\gamma}},\ u<x<\infty,$$

*whereby* $\hat{\gamma},\hat{\beta}$ *are suitable estimators for* $\gamma$ *and* $\beta$ *respectively.*

$\hat{\gamma},\hat{\beta}$ can be, for example, calculated as maximum likelihood estimators from the excesses $Y_1,\ldots,Y_{N(u)}$. First let's consider the case where $\mathrm{N}(u)=m$ is a constant and where $Y_1,\ldots,Y_m$ is a sample of i.i.d. random variables with the distribution $W_{\gamma,\beta},\gamma>0$,. Thus $W_{\gamma,\beta}$ is literally a Pareto distribution and has the probability density

$$p(y) = \frac{1}{\beta}(1+\frac{\gamma y}{\beta})^{-\frac{1}{\gamma}-1},\ \ x\geq 0.$$

Therefore, the log likelihood function is

$$\ell(\gamma, \beta \mid Y_1, \ldots, Y_m) = -m \log \beta - \left(\frac{1}{\gamma} + 1\right) \sum_{j=1}^{m} \log(1 + \frac{\gamma}{\beta} Y_j).$$

By maximising this function with respect to $\gamma, \beta$ we obtain the maximum likelihood (ML) estimator $\hat{\gamma}, \hat{\beta}$. Analogously we could also define the ML estimator for the parameter of the generalised Pareto distribution using $\gamma \leq 0$.

**Theorem 18.7**
*For all $\gamma > -\frac{1}{2}$ it holds for $m \to \infty$*

$$\sqrt{m}(\hat{\gamma} - \gamma, \ \frac{\hat{\beta}}{\beta} - 1) \xrightarrow{\mathcal{L}} N_2(0, D^{-1}),$$

*with* $D = (1+\gamma) \begin{pmatrix} 1+\gamma & -1 \\ -1 & 2 \end{pmatrix}$, *i.e.* $(\hat{\gamma}, \hat{\beta})$ *are asymptotically normally distributed. In addition they are asymptotically efficient estimators.*

In our initial problem $m = \mathrm{N}(u)$ was random. Here the estimators we have just defined, $\hat{\gamma}$ and $\hat{\beta}$, are the conditional ML estimators given $\mathrm{N}(u)$. The asymptotic distribution theory is also known in this case; in order to avoid an asymptotic bias, $\overline{F}$ must fulfill an additional regularity condition. After we find an estimator for the exceedance probability and thus a cdf for large $x$, we immediately obtain an estimator for the extreme quantile.

**Definition 18.10 (POT Quantile Estimator)**
*The* POT Quantile estimator $\hat{x}_q$ *for the q-quantile $x_q = F^{-1}(q)$ is the solution to $\overline{F}^{\wedge}(\hat{x}_q) = 1 - q$, i.e.*

$$\hat{x}_q = u + \frac{\hat{\beta}}{\hat{\gamma}} \left[ \left\{ \frac{n}{N(u)} (1-q) \right\}^{-\hat{\gamma}} - 1 \right].$$

*SFEpotquantile*

We can compare these estimators with the usual sample quantiles. To do this we select a threshold value $u$ so that exactly $k$ excesses lie beyond $u$, that is $\mathrm{N}(u) = k > n(1-q)$ and thus $u = X^{(k+1)}$. The POT quantile estimator that is dependent on the choice of $u$ respectively $k$ is

$$\hat{x}_{q,k} = X^{(k+1)} + \frac{\hat{\beta}_k}{\hat{\gamma}_k} \left[ \left\{ \frac{n}{k} (1-q) \right\}^{-\hat{\gamma}_k} - 1 \right],$$

where $\hat{\gamma}_k, \hat{\beta}_k$ is the ML estimator, dependent on the choice of $k$, for $\gamma$ and $\beta$. The corresponding sample quantile is

$$\hat{x}_q^s = X^{([n(1-q)]+1)}.$$

This is in approximate agreement with $\hat{x}_{q,k}$ when the minimal value $k = [n(1-q)]+1$ is chosen for $k$. Simulation studies show that the value $k_0$ of $k$, which minimises the mean squared error $\mathrm{MSE}(\hat{x}_{q,k}) = \mathsf{E}(\hat{x}_{q,k} - x_q)^2$, is much larger than $[n(1-q)]+1$, i.e., the POT estimator for $x_q$ differs distinctly from the sample quantile $\hat{x}_q^s$ and is superior to it with respect to the mean squared error when the thresholds $u$ or $k$ are respectively chosen.

We are interested in threshold $u$, for which the mean squared error of $\hat{x}_q$ is as small as possible. The error can be split into the variance and the squared bias of $\hat{x}_q$:

$$\mathrm{MSE}(\hat{x}_q) = \mathsf{E}(\hat{x}_q - x_q)^2 = \mathrm{Var}(\hat{x}_q) + \{\mathsf{E}(\hat{x}_q) - x_q\}^2.$$

Unfortunately the two components of the mean squared error move in opposite directions when we vary the threshold $u$ used in calculating the POT quantile estimators. We are therefore confronted with the following bias variance dilemma:

- when $u$ is too large, there are few excesses $Y_l$, $l \leq \mathrm{N}(u)$, and the estimator's variance is too large,
- when $u$ is too small, the approximation of the excess distribution using a generalised Pareto distribution is not good enough, and the bias $\mathsf{E}(\hat{x}_q) - x_q$ is no longer reliable.

An essential aid in selecting an appropriate threshold $u$ is the average excess plot, which is approximately linear beyond the appropriate threshold. This has already been discussed in Theorem 18.5, when one considers the relationship between the Fréchet distribution as the asymptotic distribution of the maxima and the Pareto distribution as the asymptotic distribution of the excesses. It is supported by the following result for the Pareto and exponential distributions $W_{\gamma,\beta}, \gamma \geq 0$,.

**Theorem 18.8**
*Let $Z$ be a $W_{\gamma,\beta}$ distributed random variable with $0 \leq \gamma < 1$. The average excess function is linear:*

$$e(u) = \mathsf{E}\{Z - u | Z > u\} = \frac{\beta + \gamma u}{1+\gamma}, \;\; u \geq 0, \;\; \textit{for } 0 \leq \gamma < 1.$$

*With the usual parametrization of the Pareto distribution $\gamma = \frac{1}{\alpha}$, i.e., the condition $\gamma < 1$ means that $\alpha > 1$ and thus $\mathsf{E}|Z| < \infty$.*

This result motivates the following application in choosing the threshold: select the threshold $u$ of the POT estimator so that the empirical average excess function $e_n(v)$ for values $v \geq u$ is approximately linear. An appropriate $u$ is chosen by considering the average excess plots, where it is recommended that the largest points $(X_{(k)}, e_n(X_{(k)}))$, $k \approx n$, along the righthand edge of the plot be excluded, since their large variability for the most part distorts the optical impression.

### 18.2.2 The Hill Estimator

The POT method for estimating the exceedance probability and the extreme quantiles can be used on data with cdf that is in the MDA of a Gumbel or a Fréchet distribution, as long as the expected value is finite. Even for extreme financial data, this estimator seems reasonable based on empirical evidence. A classic alternative to the POT method is the Hill estimator, which has already been discussed in Chapter 13 in connection with the estimation of the tail exponents of the DAX stocks. It is of course only useful for distributions with slowly decaying tails, such as those in the MDA of the Fréchet distribution, and in simulations often performs worse in comparison to the POT estimator. The details are briefly introduced in this section.

In this section we will always assume that the data $X_1, \ldots, X_n$ are i.i.d. with a distribution function $F$ in the MDA of $G_{1,\alpha}$ for some $\alpha > 0$. Due to Theorem 18.3 this is the case when $\overline{F}(x) = x^{-\alpha}L(x)$ with a slowly varying function $L$. The tapering behaviour of $\overline{F}(x) = \mathrm{P}(X_t > x)$ for increasing $x$ is mainly determined by the so called *tail exponents* $\alpha$. The starting point of the Hill method is the following estimator for $\alpha$.

**Definition 18.11 (Hill estimator)**
$X^{(1)} \geq X^{(2)} \geq \ldots \geq X^{(n)}$ *are the order statistics in decreasing order. The* Hill estimator $\hat{\alpha}_H$ *of the tail exponents* $\alpha$ *for a suitable* $k = k(n)$ *is*

$$\hat{\alpha}_H = \{\frac{1}{k}\sum_{j=1}^{k} \log\ X^{(j)} - \log\ X^{(k)}\}^{-1}.$$

The form of the estimator can be seen from the following simple special case. In general it holds that $\overline{F}(x) = L(x)/(x^{\alpha})$, but we now assume that with a fixed $c > 0$ $L(x) = c^{\alpha}$ is constant. Set $V_j = \log(X_j/c)$, it holds that

$$\mathrm{P}(V_j > v) = \mathrm{P}(X_j > ce^{v}) = \overline{F}(ce^{v}) = \frac{c^{\alpha}}{(ce^{v})^{\alpha}} = e^{-\alpha v},\ y \geq 0,$$

$V_1, \ldots, V_n$ are therefore independent exponentially distributed random variables with parameter $\alpha$. As is well known it holds that $\alpha = (\mathsf{E}V_j)^{-1}$, and the ML estimator $\hat{\alpha}$ for $\alpha$ is $1/\overline{V}_n$, where $\overline{V}_n$ stands for the sample average of $V_1, \ldots, V_n$, thus,

$$\hat{\alpha} = \frac{1}{\overline{V}_n} = \{\frac{1}{n}\sum_{j=1}^{n} \log(X_j/c)\}^{-1} = \{\frac{1}{n}\sum_{j=1}^{n} \log\ X^{(j)} - \log\ c\}^{-1},$$

where for the last equation only the order of addition was changed. $\hat{\alpha}$ is already similar to the Hill estimator. In general it of course only holds that $\overline{F}(x) \approx \frac{c^\alpha}{x^\alpha}$ for sufficiently large $x$. The argument for the special case is similar for the largest observations $X^{(1)} \geq X^{(2)} \geq \ldots \geq X^{(k)} \geq u$ beyond the threshold $u$, so that only the $k$ largest order statistics enter the definition of the Hill estimator.

The Hill estimator is consistent, that is it converges in probability to $\alpha$ when $n, k \to \infty$ such that $k/n \to 0$. Under an additional condition it can also be shown that $\sqrt{k}(\hat{\alpha}_H - \alpha) \xrightarrow{\mathcal{L}} \mathrm{N}(0, \alpha^2)$, i.e., $\hat{\alpha}_H$ is asymptotically normally distributed.

Similar to the POT estimator when considering the Hill estimator the question regarding the choice of the threshold $u = X^{(k)}$ comes into play, since the observations located beyond it enter the estimation. Once again we have a bias variance dilemma:

- When $k$ is too small, only a few observations influence $\hat{\alpha}_H$, and the variance of the estimator, which is $\alpha^2/k$ asymptotically, is too large,
- when $k$ is too large, the assumption underlying the derivation of the estimator, i.e., that $L(x)$ is approximately constant for all $x \geq X^{(k)}$, is in general not well met and the bias $\mathsf{E}\hat{\alpha}_H - \alpha$ becomes too large.

Based on the fundamentals of the Hill estimator for the tail exponents $\alpha$ we obtain direct estimators for the exceedance probability $\overline{F}(x)$ and for the quantiles of $F$. Since $\overline{F}(x) = x^{-\alpha}L(x)$ with a slowly varying function $L$, it holds for large $x \geq X^{(k)}$ that:

$$\frac{\overline{F}(x)}{\overline{F}(X^{(k)})} = \frac{L(x)}{L(X^{(k)})}\left(\frac{X^{(k)}}{x}\right)^{\alpha} \approx \left(\frac{X^{(k)}}{x}\right)^{\alpha}, \tag{18.5}$$

Because exactly one portion $k/n$ of the data is larger or equal to the order statistic $X^{(k)}$, this is the $(1 - k/n)$ sample quantile. Therefore, the empirical distribution function takes on the value $1 - k/n$ at $X^{(k)}$, since it uniformly converges to the distribution function $F$, for sufficiently large $n$, a $k$ that is not too large in comparison to $n$ yields: $F(X^{(k)}) \approx 1 - k/n$, i.e., $\overline{F}(X^{(k)}) \approx k/n$.

Substituting this into (18.5), we obtain a *Hill esitmator for the exceedance probability* $\overline{F}(x)$ :

$$\overline{F}_H^{\wedge}(x) = \frac{k}{n}\left(\frac{X^{(k)}}{x}\right)^{\hat{\alpha}_H}$$

By inverting this estimator we have the *Hill quantile estimator* for the $q-$ quantile $x_q$ with $q \approx 1$ :

$$\begin{aligned}\hat{x}_{q,H} &= X^{(k)}\left\{\frac{n}{k}(1-q)\right\}^{-1/\hat{\alpha}_H} \\ &= X^{(k)} + X^{(k)}\left[\left\{\frac{n}{k}(1-q)\right\}^{-\hat{\gamma}_H} - 1\right]\end{aligned}$$

with $\hat{\gamma}_H = 1/\hat{\alpha}_H$, where the second representation clearly shows the similarities and differences to the POT quantile estimator.

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## 18.3 Estimators for Risk Measurements

The value at risk discussed in the previous chapter is not the single measure of the market risk. In this section we introduce an alternative risk measure. In addition we discuss how to estimate the measure given extremely high loss.

**Definition 18.12 (Value-at-Risk and Expected Shortfall)**
*Let* $0 < q < 1$, *and let* $F$ *be the distribution of the loss* $X$ *of a financial investment within a given time period, for example, one day or 10 trading days. Typical values for* $q$ *are* $q = 0.95$ *and* $q = 0.99$.
*a) The* Value-at-Risk *(VaR) is the q-quantile*

$$VaR_q(X) = x_q = F^{-1}(q).$$

*b) The* expected shortfall *is defined as*

$$S_q = \mathsf{E}\{X | X > x_q\}.$$

Value-at-Risk is today still the most commonly used measurement, which can quantify the market risk. It can be assumed, however, that in the future the expected shortfall will play at least an equal role.

**Definition 18.13 (Coherent Risk Measure)**
*A* coherent risk measure *is a real-valued function* $\rho : \mathbb{R} \to \mathbb{R}$ *of real-valued*

*random variables, which model the losses, with the following characteristics:*

| | | |
|---|---|---|
| *(A1)* | $X \geq Y$ *a.s.* $\Longrightarrow \rho(X) \geq \rho(Y)$ | *(Monotonicity)* |
| *(A2)* | $\rho(X+Y) \leq \rho(X) + \rho(Y)$ | *(Subadditivity)* |
| *(A3)* | $\rho(\lambda X) = \lambda\rho(X)$ *for* $\lambda \geq 0$ | *(Positive homogeneity)* |
| *(A4)* | $\rho(X+a) = \rho(X) + a$ | *(Translation equivariance)* |

These conditions correspond to intuitive obvious requirements of a market risk measurement:
(A1) When the loss from investment $X$ is always larger than that from investment $Y$, then the risk from investment $X$ is also larger.
(A2) The risk of a portfolio consisting of investments in $X$ and $Y$ is at most as large as the sum of the individual risks (diversification of the risk).
(A3) When an investment is multiplied, then the risk is also multiplied accordingly.
(A4) By adding a risk free investment, i.e., a non-random investment with known losses $a$ ($a < 0$, when the investment has fixed payments), to a portfolio, the risk changes by exactly $a$.

The VaR does not meet condition (A2) in certain situations. Let $X$ and $Y$, for example, be i.i.d. and both can take on the value 0 or 100 with probability $\mathrm{P}(X = 0) = \mathrm{P}(Y = 0) = p$ and $\mathrm{P}(X = 100) = \mathrm{P}(Y = 100) = 1 - p$. Then $X + Y$ can be 0, 100 and 200 with probability $\mathrm{P}(X + Y = 0) = p^2$, $\mathrm{P}(X + Y = 100) = 2p(1-p)$ and $\mathrm{P}(X + Y = 200) = (1-p)^2$ respectively. For $p^2 < q < p$ and $q < 1 - (1-p)^2$, for example, for $p = 0.96, q = 0.95$, it holds that

$$VaR_q(X) = VaR_q(Y) = 0, \quad \text{but} \quad VaR_q(X+Y) = 100.$$

The expected shortfall, on the other hand, is a coherent risk measure that always fulfills all four conditions. It also gives a more intuitive view of the actual risk of extreme losses than the Value-at-Risk. The VaR only depends on the probability of losses above the $q$-quantile $x_q$, but it doesn't say anything about whether these losses are always just a little above the threshold $x_q$ or whether there are also losses that are much larger than $x_q$ that need to be taken into account. In contrast the expected shortfall is the expected value of the potential losses from $x_q$ and depends on the actual size of the losses.

The Value-at-Risk is simply a quantile and can be, for example, estimated as a sample quantile $\hat{F}_n^{-1}(q)$, where $\hat{F}_n(x)$ is the empirical distribution of a

sample of negative values, i.e., losses, from the past. As was discussed at the beginning of the chapter, this particular estimator of $q \approx 1$, which is for the typical VaR-level of 0.95 and 0.99, is often too optimistic. An alternative VaR estimator, which has the possibility of reflecting extreme losses better, is the POT or the Hill quantile estimator.

Analogous estimators for the expected shortfall are easy to derive. This risk measure is closely related to the average excess function when $u = x_q$, as immediately can be seen from the definition:

$$S_q = e(x_q) + x_q.$$

Here we only consider the POT estimator for $S_q$. Since $F_u(x) \approx W_{\gamma,\beta}(x)$ for a sufficiently large threshold $u$, it holds from Theorem 18.5, b) with $\alpha = 1/\gamma$

$$e(v) \approx \frac{\beta + (v-u)\gamma}{1-\gamma} \quad \text{for} \quad v > u.$$

Therefore, for $x_q > u$ we have

$$\frac{S_q}{x_q} = 1 + \frac{e(x_q)}{x_q} \approx \frac{1}{1-\gamma} + \frac{\beta - \gamma u}{x_q(1-\gamma)}.$$

The *POT estimator for the expected shortfall* $S_q$ is thus

$$\hat{S}_{q,u} = \frac{\hat{x}_q}{1-\hat{\gamma}} + \frac{\hat{\beta} - \hat{\gamma} u}{1-\hat{\gamma}},$$

where $\hat{x}_q$ is the POT quantile estimator.

## 18.4 Extreme Value Theory for Time Series

Let $Z_t$, $-\infty < t < \infty$, be a *strictly stationary time series*, as defined in Definition 11.6, that is the distribution of the data and its probability structure does not change over time. Each single observation $Z_t$ has, among other things, the same distribution function $F$. To compare consider the i.i.d. random variables $X_1, X_2, \ldots$ with the same distribution $F$. Let $M_n = \max\{Z_1, \ldots, Z_n\}$, $M_n^x = \max\{X_1, \ldots, X_n\}$ be the maxima of $n$ values from the time series respectively from $n$ independent observations. A simple but basic relationship for the previous sections is (18.1), i.e.,

$$\mathrm{P}(M_n^x \le y) = \{\mathrm{P}(X_j \le y)\}^n = F^n(y),$$

where the independence of $X_t$ is used. For dependent data this relationship does not hold and the distribution of the maximum $M_n$ is not determined by

$F$ alone, but rather from the complete distribution of the time series. Luckily in many cases there is at least one comparable, approximate relationship:

$$\mathrm{P}(M_n \leq y) \approx F^{n\delta}(y) \geq F^n(y) \quad \text{for large } n,$$

where $\delta \in [0,1]$ is the so called extremal index. In order to find an exact definition, recall Theorem 18.2 for the independent case, whereby

$$\begin{aligned} & n\overline{F}(u_n) \to \tau \\ \text{if and only if} \quad & \mathrm{P}(M_n^x \leq u_n) \to e^{-\tau}. \end{aligned}$$

**Definition 18.14 (Extremal Index)**
$\delta \in [0,1]$ *is called the* extremal index *of the time series* $Z_j$, $-\infty < j < \infty$, *when for certain* $\tau, u_n$

$$n\overline{F}(u_n) \to \tau \quad \textit{and} \quad \mathrm{P}(M_n \leq u_n) \to e^{-\delta\tau}.$$

*(If* $\delta$ *exists, then the value does not depend on the specific choice of* $\tau, u_n$*).*

From the definition the above claimed approximate relationship between the distribution of the maximum and the exceedance probability immediately follows:

$$\mathrm{P}(M_n \leq u_n) \approx e^{-\delta\tau} \approx e^{-\delta n\overline{F}(u_n)} = (e^{-\overline{F}(u_n)})^{n\delta} \approx (1-\overline{F}(u_n))^{n\delta} = F^{n\delta}(u_n),$$

when $u_n$ is large and thus $\overline{F}(u_n) \approx 0$.

Pure white noise automatically has the extremal index $\delta = 1$, since $Z_t$ here are independent. It is not obvious that all ARMA$(p,q)$ processes (see Chapter 12) with normally distributed innovations also have an extremal index $\delta = 1$, its maxima thus behave like maxima from independent data. Intuitively this comes from, on the one hand, ARMA processes having an exponentially decreasing memory, i.e., the observations $Z_t, Z_{t+\tau}$ are for sufficiently large time periods $\tau$ practically independent, and, on the other hand, the probability of two extreme observations occurring within the same time interval (which is not too long) is low. These qualitative statements can be formulated as two precise criteria of time series that have an extremal index of 1, the exact formulation of which will not be given here.

For financial time series models the second condition is not fulfilled, because they contradict the presence of volatility clusters (see Chapter 13), i.e., the local frequency of extreme observations. The extremal index of an ARCH(1) process with parameters $\omega, \alpha$ (see Definition 13.1) is, for example, always $\delta = \delta(\alpha) < 1$. It can be approximated for $\alpha = 0.5$, for example, $\delta \approx 0.835$.

Finally note that not every time series has an extremal index. A simple counter example is $Z_t = A \cdot X_t$ with i.i.d. random variables $X_t$, which are modelled by a random factor $A > 0$ that is independent of $X_t$. Since the factor $A$ is contained in all observations, even in the most distant past, this time series has no decreasing memory. If the distribution of $X_t$ has slowly decaying tails, i.e., they belong to the MDA of a Fréchet distribution, then it can be shown that $Z_t$ can not have an extremal index.

The extreme theory for time series is still developing. The Fisher-Tippett theorem, however, exists as a central result in the following modified form:

**Theorem 18.9**
*Let $\{Z_t\}$ be a strictly stationary time series with the distribution function $F$ and an extremal index $\delta > 0$. Let $X_1, X_2, \ldots$ be i.i.d. with the same distribution function $F$. $M_n^x = max\{X_1, \ldots, X_n\}$. Let $G_\gamma$ be a general extreme value distribution. We have*

$$\begin{aligned} & \mathrm{P}\left(\tfrac{M_n^x - d_n}{c_n} \le x\right) \to G_\gamma(x) \\ \textit{if and only if} \quad & \mathrm{P}\left(\tfrac{M_n - d_n}{c_n} \le x\right) \to G_\gamma^\delta(x) \end{aligned}$$

*for all $x$ with $0 < G_\gamma(x) < 1$.*

The maxima of the time series are standardised by the same series $c_n, d_n$ and converge in distribution to the same type of asymptotic distribution as the maxima of the corresponding independent data, since $G_\gamma^\delta$ is itself a general extreme value distribution with the same form parameters as $G_\gamma$. For example, for $\gamma > 0$ it holds that

$$G_\gamma^\delta(x) = \exp\{-\delta(1+\gamma x)^{-1/\gamma}\} = G_\gamma(\frac{x-\mu}{\sigma}), \;\; 1 + \gamma x > 0$$

with $\sigma = \delta^\gamma$ and $\mu = -(1 - \delta^\gamma)$, i.e., except for the location and scale parameters the distributions are identical.

Many of the techniques used in extreme value statistics, that were developed for independent data can be used on time series. To do this, however, one needs to have more data, because the effective size of the sample is only $n\delta$ instead of $n$. Besides that, additional problems appear: the POT method is perhaps in theory still applicable, but the excesses are no longer independent, especially when a financial time series with volatility clusters is considered. For this reason the parameters of the generalised Pareto distribution, with which the excess distribution is approximated, cannot be estimated by simply taking the maximum of the likelihood function of independent data. One way out of this is to either use special model assumptions, with which the

likelihood function of the dependent excesses can be calculated, or by using a reduction technique, with which the data is made more "independent" at the cost of the sample size. One application, for example, replaces the cluster of neighbouring excesses with a maximum value from the cluster, whereby the cluster size is so chosen that the sample size of the excesses is approximately reduced by the factor $\delta$. Afterwards the POT estimators, which were developed for independent data, can be calculated from the reduced excesses.

Another problem is that the extremal index needs to be estimated in order to be able to use applications like the one just described. In the literature several estimation techniques are described. We will introduce only one here; one that can be described without a lot of technical preparation, the so called *Block method*. First the time series data $Z_1, \ldots, Z_n$ is divided into $b$ blocks, each has a length $l$ (size $n = bl$, $b$, $l$ large). Let $M_l^{(k)}$ be the maximum of the observations in the $k$-th block:

$$M_l^{(k)} = \max(Z_{(k-1)l+1}, \ldots, Z_{kl}),\ k = 1, \ldots, b.$$

For a large threshold value $u$, let $\mathrm{N}(u) = \#\{t \leq n;\ Z_t > u\}$ be the number of observations beyond the threshold and let $B(u) = \#\{k \leq b;\ M_l^{(k)} > u\}$ be the number of blocks with at least one observation beyond the threshold $u$. The estimator for the extremal index is then

$$\hat{\delta} = \frac{1}{l} \frac{\log\,(1 - \frac{B(u)}{b})}{\log\,(1 - \frac{\mathrm{N}(u)}{n})}.$$

Heuristically this estimator can be derived from the following three observations:

(i) From the definition of the extremal index it follows that $\mathrm{P}(M_n \leq u) \approx F^{\delta n}(u)$, when $n, u \to \infty$, so that $n\overline{F}(u) \to \tau$. Solving for $\delta$ it follows that

$$\delta \approx \frac{\log\ \mathrm{P}(M_n \leq u)}{n \log\ F(u)}.$$

(ii) $F$ can be estimated using the empirical distribution function $\hat{F}_n$, so that $F(u) = 1 - \mathrm{P}(Z_t > u) \approx 1 - \frac{\mathrm{N}(u)}{n}$ .

(iii) With $n = bl$ it follows that

$$\begin{aligned} \mathrm{P}(M_n \leq u) &\approx \prod_{k=1}^{b} \mathrm{P}(M_l^{(k)} \leq u) \approx \{\mathrm{P}(M_l^{(1)} \leq u)\}^b \\ &\approx \{\frac{1}{b} \sum_{k=1}^{b} \mathbf{1}(M_l^{(k)} \leq u)\}^b = (1 - \frac{B(u)}{b})^b. \end{aligned}$$

By combining the three observations we have

$$\delta \approx \frac{b \ \log\ (1 - \frac{B(u)}{b})}{n \ \log\ (1 - \frac{\mathrm{N}(u)}{n})} = \hat{\delta}.$$

## 18.5 Recommended Literature

Both of the basic theorems, Theorem 18.1 and Theorem 18.6, of this section go back to Fisher and Tippett (1928) respectively Pickands (1975). The essential notion of quantifying risk by coherent risk measures was introduced by Artzner and Heath (1997).

A comprehensive summary of the modelling and statistical analysis of extreme results is given in the monograph from Embrechts et al. (1997). There one finds proofs as well as detailed mathematical and practical considerations of the content of this section and an extensive bibliography. Another actual and recommendable book on extreme value statistic is Reiss and Thomas (1997). A more in depth implementation of the method in the form of quantlets discussed in this last reference, which goes beyond the selection introduced in this section, can be found in Reiss and Thomas (2000).

A substantial problem that occurs when applying the methods of extreme value statistics such as the POT or the Hill estimators is the choice of the threshold value $u$ or the corresponding number $k$ of large order statistics. We have already mentioned how this choice can be made with the help of graphical representations. A more in depth discussion including the corresponding quantlet can be found in Reiss and Thomas (2000). Polzehl and Spokoiny (2003) and Grama and Spokoiny (2003) describe current procedures used for estimating the tail exponents, for which the choice of $u$ or $k$ respectively, given the available data, can be adaptively and thus automatically chosen.

The methods described in this chapter give estimators for the Value-at-Risk as unconditional quantiles. Often one wishes to include financial data from the recent past when estimating risk, for example in a GARCH(1,1) model the last observation and the last volatility. In this case the Value-at-Risk is a conditional quantile given the available information. One possibility of using extreme value statistics in such cases is based on the assumptions of a specific stochastic volatility model which is parametric as in McNeil and Frey (2000) or nonparametric as in Chapter 14.

Given the assumptions of the model a conditional volatility $\sigma_t$ is estimated given the past, which together with the data results in an estimator for the innovations $Z_t$. In calculating the conditional quantile it is not assumed that

the $Z_t$ are standard normally distributed, but instead the needed unconditional quantile of the innovations is estimated from the estimated innovations with, for example, the POT estimator. Alternatively one can estimate the conditional quantile also direct as nonparametric, in which the conditional distribution function is first estimated with a kernel estimator and then the inverse is taken. With moderately large quantiles, for example, with a 95% VaR, the method from Franke and Mwita (2003) gives good results, even for innovation distributions with heavy tails and infinite variance. For extreme quantiles such as the 99% VaR a semi-parametric method must be considered, as is the case with the POT method, in order to obtain useful results. Mwita (2003) estimates first a nonparametric, medium-sized conditional quantile and modifies this estimator through the fitting of a Pareto distribution to the extreme excesses.

# 19 Neural Networks

A neural network is a non-linear system that converts a series of real input values $x_1, \ldots, x_p$ over several intermediary steps to one or more terminal variables $y_1, \ldots, y_q$. It represents a function $\nu : \mathbb{R}^p \to \mathbb{R}^q$:

$$(y_1, \ldots, y_q)^\top = \nu(x_1, \ldots, x_p),$$

that has a special form given by the network structure. This is graphically

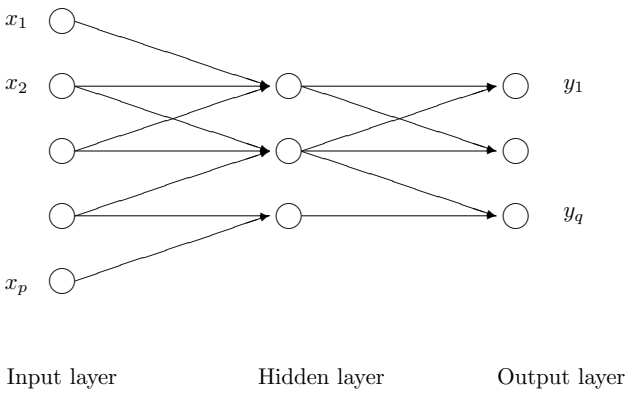


Figure 19.1: Neural feed forward network with a hidden layer

displayed in Figure 19.1 in the form of a directed graph whose nodes are grouped in various levels. In the *input layer* each node represents an input variable; in the *output layer* each node represents an output variable. In between there are one or more *hidden layers*, whose nodes are neither sources nor layers of the graph. The network in Figure 19.1 contains only one hidden layer. In additional it is a *feed forward network*, since it contains no edges that begin in a node and end in the same node or in a different node from the same or a previous layer.

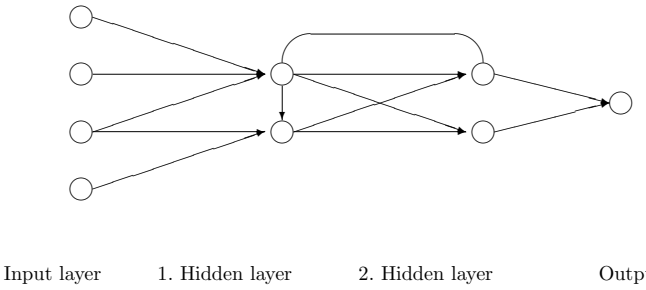


Figure 19.2: Neural feedback network with two hidden layers

Figure 19.2 displays a *feedback network*, in that there is feedback among the nodes of the two hidden layers. In the following we will concentrate on the feed forward network.

Neural networks are used in financial statistics to represent functions, which, for example, can represent the default probability of a credit, the forecast of an exchange rate or the volatility of a stock. Here the emphasis is on non-parametric applications, which in comparison to the local smoothing function discussed in Chapter 14 require advanced modelling and can be quite involved to calculate. On the other hand it is still practical when numerous variables need to be considered in forecasts or quantifying risk, i.e., when the dimension $p$ of the function arguments is large.

Since neural networks are still relatively unknown tools in statistics, in the first section we will give an elementary introduction in the structure of a neural network. It allows for the construction of complex functions using simple elements. In the second section we describe the popular numerical application for fitting neural networks to the data, before we conclude with various applications to financial problems and introduce the underlying assumptions.

## 19.1 From Perceptron to Non-linear Neuron

The perceptron is a simple mathematical model of how a nerve cell functions in receiving signals from sense cells and other nerve cells (the input variables) and from this sends a signal to the next nerve cell or remains inactive. In spite of all of the disadvantages the perceptron is very influential on the way

of thinking with respect to neural networks, so that it is a good starting point for the discussion of components from which neural networks are constructed. The perceptron works in two steps:

– the input variables $x_1, \ldots, x_p$ are multiplied and added with *weights* $w_1, \ldots, w_p$,

– a *threshold operation* is applied to the result.

$x = (x_1, \ldots, x_p)^\top$, $w = (w_1, \ldots, w_p)^\top$ represent the input vector and weight vector respectively, and for a given $b$ let $\psi(u) = \mathbf{1}(u > b)$ be the corresponding threshold function. The output variables $y = \psi(w^\top x)$ of the perceptron is 1 (the nerve cell "fires"), when the sum of the weighted input signals lies above the threshold and is 0 otherwise (the nerve cells remain inactive). The effect

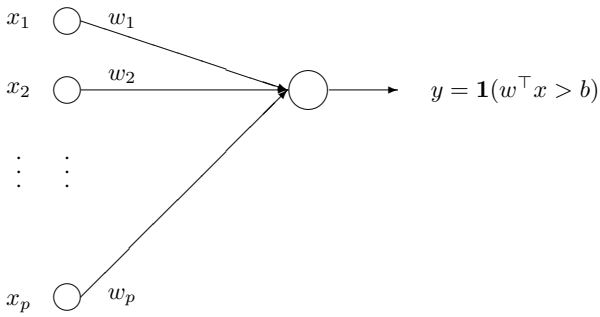


Figure 19.3: The perceptron

of the perceptron depends on the weights $w_1, \ldots, w_p$ and the threshold value $b$. An equivalent representation can be obtained by including the constant $x_0 \stackrel{\text{def}}{=} 1$ as an additional input variable, which has the weight $w_0 = -b$ and a threshold value of 0 is chosen since then

$$\mathbf{1}\left(\sum_{i=1}^{p} w_i x_i > b\right) = \mathbf{1}\left(\sum_{i=0}^{p} w_i x_i > 0\right).$$

This representation is often more comfortable since with the system parameters that can be freely chosen one does not have to differentiate between weights and threshold values.

A perceptron can be trained to solve classification problems of the following type: Given are objects which belong to one of two classes, $C_0$ or $C_1$. De-

cisions are made based on observations of the object $x_1, \ldots, x_p$, whether it belongs to $C_0$ or $C_1$.

The perceptron characterised by the weights $w_0, \ldots, w_p$ classifies an object as belonging to $C_0$ respectively $C_1$ when the output variable $y = y(x_1, \ldots, x_p)$ is 0 respectively 1. So that the classification problem "may be" solved, the weights $w_0, \ldots, w_p$ must be "learned". To do this there is a *training set*

$$(x^{(1)}, z^{(1)}), \ldots, (x^{(T)}, z^{(T)})$$

from $T$ input vectors

$$x^{(t)} = (x_1^{(t)}, \ldots, x_p^{(t)})^\top$$

available whose correct classification

$$z^{(1)}, \ldots, z^{(T)} \in \{0, 1\}$$

is known. With the help from *learning rules* suitable weights $\hat{w}_0, \ldots, \hat{w}_p$ are determined from the training set.

In statistical terms the problem is to estimate the parameters of the perceptron from the data $(x^{(t)}, z^{(t)})$, $t = 1, \ldots, T$. A *learning rule* is an estimation method which produces estimates $\hat{w}_0, \ldots, \hat{w}_p$.

A learning rule is, for example, the Delta or *Widrow-Hoff learning rule*: The input vectors $x^{(t)}, t = 1, \ldots, T$, are used consecutively as input variables of the perceptron and the output variables $y^{(t)},\ t = 1, \ldots, T$, are compared to the correct classification $z^{(t)},\ t = 1, \ldots, T$. If in one step $y^{(t)} = z^{(t)}$, then the weights remain unchanged. If on the other hand $y^{(t)} \neq z^{(t)}$, then the weight vector $w = (w_0, \ldots, w_p)^\top$ is adjusted in the following manner:

$$w_{new} = w + \eta(z^{(t)} - y^{(t)})\, x^{(t)}$$

$\eta$ is a small relaxation factor which must eventually slowly approach zero in order to ensure convergence of the learning algorithm. The initial value of $w$ is arbitrarily given or randomly chosen, for example, uniformly distributed over $[0, 1]^{p+1}$.

The learning does not end when all of the input vectors are presented in the network, but rather after $x^{(T)}$ has been entered, $x^{(1)}$ is used again as the next input variable. The training set is tested multiple times until the network of all objects in the training set have been correctly identified or until a given quality criterion for measuring the error in classification is low enough.

The weights $w_0, \ldots, w_p$ can be identified up to a positive scale factor, i.e., for $\alpha > 0$, $\alpha\, w_0, \ldots, \alpha\, w_p$ lead to the same classification. By applying the learning rule, for example, the Widrow-Hoff, it is possible that $||w||$ will continuously increases; this can lead to numerical problems. In order to prohibit this, one uses the so called *weight decay* technique, i.e., a modified learning rule in which $||w||$ remains stable.

**Example 19.1 (Learning the OR-Function)**
*Let $p = 2$ and $x_1, x_2 \in \{0, 1\}$. The classifications that needs to be learned is the logical OR:*

$$z = 1, \text{ if } x_1 = 1 \text{ or } x_2 = 1,$$
$$z = 0, \text{ if } x_1 = 0 \text{ and } \quad x_2 = 0.$$

*The following input vectors, including the first coordinate $x_0 = 1$*

$$x^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ x^{(2)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ x^{(3)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ x^{(4)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

*are used as the training set with the correct classification $z^{(1)} = z^{(2)} = z^{(3)} = 1,\ z^{(4)} = 0$. The perceptron with the weights $w_0, w_1, w_2$ classifies an object as 1 if and only if*

$$w_0 x_0 + w_1 x_1 + w_2 x_2 > 0\ ,$$

*and as 0 otherwise. For the starting vector we use $w = (0, 0, 0)^\top$, and we set $\eta = 1$. The individual steps of the Widrow-Hoff learning take the following form:*

1. *$w^{(1)}$ gives $y^{(1)} = 0 \neq z^{(1)}$. The weights are changed:*
   $w_{new} = (0, 0, 0)^\top + (1 - 0)(1, 1, 0)^\top = (1, 1, 0)^\top$
2. *$x^{(2)}$ is correctly classified with the weight vector.*
3. *$x^{(3)}$ is correctly classified with the weight vector.*
4. *For $x^{(4)}$ is $w^\top x^{(4)} = 1$, so that the weights are again changed:*
   $w_{new} = (1, 1, 0)^\top + (0 - 1)(1, 0, 0)^\top = (0, 1, 0)^\top$
5. *$x^{(1)}$ is now used as input and is correctly classified.*
6. *Since $w^\top x^{(2)} = 0$ :*
   $w_{new} = (0, 1, 0)^\top + (1 - 0)(1, 0, 1)^\top = (1, 1, 1)^\top$
7. *Since $w^\top x^{(3)} = 3 > 0$, $x^{(3)}$ is correctly classified.*
8. *$x^{(4)}$ is incorrectly classified so that*
   $w_{new} = (1, 1, 1)^\top + (0 - 1)(1, 0, 0)^\top = (0, 1, 1)^\top$

*Thus the procedure ends since the perceptron has correctly identified all the input vectors in the training set with these weights. The perceptron learned the OR function over the set* $\{0,1\}^2$.

One distinguishes different types of learning for neural networks:

*Supervised Learning:* Compare the network outputs $y = y(x_1, \ldots, x_p)$ with the correct $z = z(x_1, \ldots, x_p)$. When $y \neq z$, the weights are changed according to the learning rule.

*Reinforcement Learning:* From every network output $y = y(x_1, \ldots, x_p)$ one discovers, whether it is "correct" or "incorrect" - in the latter case though one does not know the correct value. When $y$ is "incorrect", the weights are changed according to the learning rule.

*Unsupervised Learning:* There is no feedback while learning. Similar to the cluster analysis, random errors are filtered from the data with the help of redundant information.

For $y \in \{0,1\}$ supervised and reinforcement learning are the same. Included in this type is the Widrow-Hoff learning rule for the perceptron.

The perceptron cannot learn all of the desired classifications. The classical counter example is the logical argument XOR = "exclusive or":

$$\begin{aligned} &z = 1, \text{ if either } x_1 = 1 \text{ or } x_2 = 1, \\ &z = 0, \text{ if } x_1 = x_2 = 0 \text{ or } \quad x_1 = x_2 = 1. \end{aligned}$$

A perceptron with weights $w_0, w_1, w_2$ corresponds to a hyperplane $w_0 + w_1 x_1 + w_2 x_2 = 0$ in $\mathbb{R}^2$ space of the inputs $(x_1, x_2)^\top$, which separates the set using the perceptron of 0 classified objects from those classified as 1. It is not hard to see that no hyperplane exists for "exclusive or" where inputs should be classified as 1 $\binom{1}{0}, \binom{0}{1}$ can be separated from those to be classified as 0 $\binom{0}{0}, \binom{1}{1}$.

**Definition 19.1 (linearly separable)**
*For* $p \geq 1$ *to subsets* $\mathcal{X}_0, \mathcal{X}_1 \subseteq \mathbb{R}^p$ *are called* linearly separable *if* $w \in \mathbb{R}^p$, $w_0 \in \mathbb{R}$ *exists with*

$$\begin{aligned} w_0 + w^\top x > 0 \quad &\text{for } x \in \mathcal{X}_1, \\ w_0 + w^\top x \leq 0 \quad &\text{for } x \in \mathcal{X}_0. \end{aligned}$$

The perceptron with $p$ input variables $x_1, \ldots, x_p$ (with respect to the constant $x_0 \stackrel{\text{def}}{=} 1$) can learn the classification exactly that is consistent with the

linearly separable sets of inputs.

If no perfect classification is possible through a perceptron, then one can at least try to find a "good" classification, that is, to determine the weights $w_0, \ldots, w_p$ so that a measurement for the amount of incorrectly identified classifications can be minimised. An example of such an application is given by the *least squares (LS) classification:*

Assuming that the training set $(x^{(1)}, z^{(1)}), \ldots, (x^{(T)}, z^{(T)})$ is given. Determine for some given weight $w_0$ the weights $w_1, \ldots, w_p$ so that

$$\begin{aligned} Q(w) &= Q(w_1, \ldots, w_p) = \sum_{i=1}^{T} (z^{(i)} - y^{(i)})^2 = \min! \\ &\text{with } y^{(i)} = \mathbf{1}(w_0 + w^\top x^{(i)} > 0),\ w = (w_1, \ldots, w_p)^\top \end{aligned}$$

$w_0$ can be arbitrarily chosen since the weights $w_0, \ldots, w_p$ described above are only determined up to a scale factor. In the case of the perceptron, which takes on a binary classification, $Q(w)$ is simply the number of incorrectly defined classifications. The form mentioned above can also be directly applied to other problems. The attainable minimum of $Q(w)$ is exactly 0 (perfect classification of the training set) when both sets

$$\mathcal{X}_0^{(T)} = \{x^{(i)},\ i \le T;\ z^{(i)} = 0\},\ \mathcal{X}_1^{(T)} = \{x^{(i)},\ i \le T;\ z^{(i)} = 1\}$$

are linearly separable.

The Widrow-Hoff learning rule solves the LS classification problem; there are, however, a series of other learning rules or estimation methods which can also solve the problem. The perceptron has proven to be too inflexible for many applications. Therefore, one considers general forms of *neurons* as components used to build a neuron network:

Let $x = (x_1, \ldots, x_p)^\top,\ w = (w_1, \ldots, w_p)^\top$ be input and weight vectors respectively. For $\beta, \beta_0 \in \mathbb{R}$

$$\psi_\beta(t) = \frac{1}{1 + \exp(-\frac{t+\beta}{\beta_0})}$$

is the logistic function, which due to its form is often referred to as "the" *sigmoid function.* One can also use other functions with sigmoid forms, for example, the density function of a normal distribution. The output variable of the neuron is $y = \psi_\beta(w^\top x)$.

For $\beta_0 \to 0+$ $\psi_\beta(t)$ approaches a threshold function:

$$\psi_\beta(t) \longrightarrow \mathbf{1}(t + \beta > 0) \qquad \text{for } \beta_0 \longrightarrow 0+ \ ,$$

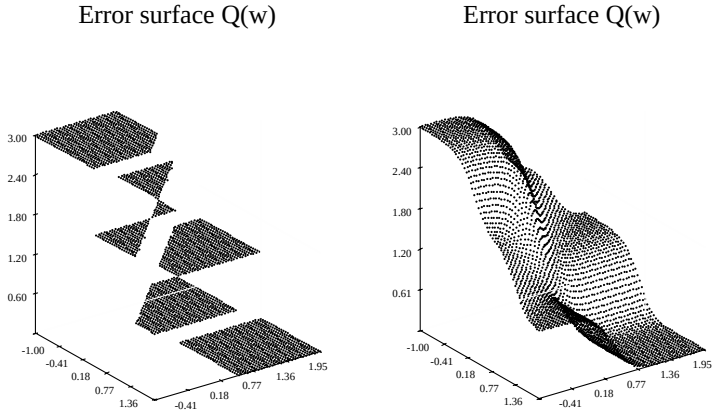


Figure 19.4: Error surface of $Q(w)$ given weight $w = (w_1, w_2)^\top$ with transform function: threshold function (left) and sigmoid function (right) SFEerrorsurf

so that the preceptron is a boundary of the neuron with a logistic activation function. An example of $Q(w)$ for neurons with threshold function and sigmoid function as activation function is shown in Figure 19.4. The corresponding method is presented in Figure 19.5.

$\beta_0$ is often not explicitly chosen, since it can be integrated as a scale factor in the other parameters $w_1, \ldots, w_p$, $\beta$ of the neurons. If one also sets $w_0 = \beta$ and $x_0 \stackrel{\text{def}}{=} 1$,, then the output variables can also be written in the form:

$$y = \psi(w_0 + w^\top x) = \psi(\sum_{k=0}^{p} w_k x_k) \text{ with } \psi(t) = \frac{1}{1 + e^{-t}}.$$

By combining multiple neurons with sigmoid or - in the limit case - threshold activation functions with a feed forward network one obtains a so called *multiple layer perceptron* (MLP) neural network. Figure 19.6 shows such a neural network with two input variables with respect to the constant $x_0 \stackrel{\text{def}}{=} 1$, two sigmoid neurons in the hidden layer that are connected by another sigmoid neuron to the output variables, where $\psi(t) = \{1 + e^{-t}\}^{-1}$ as above.

Neural networks can also be constructed with multiple hidden layers that give multiple output variables. The connections do not have to be complete, i.e.,

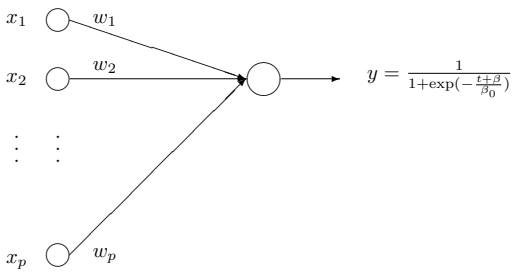


Figure 19.5: Neuron with a sigmoid transformation function

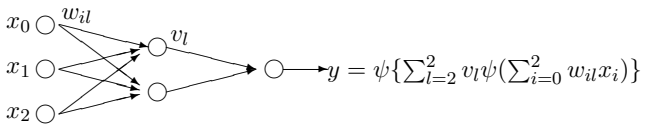


Figure 19.6: Multiple layer perceptron with a hidden layer

edges between the nodes of consecutive layers may be missing or equivalently several weights can be set to 0. Instead of the logical function or similar sigmoid functions, threshold functions may also appear in some neurons. Another probability is the so called *radial basis functions* (RBF). To the former belongs the density of the standard normal distribution and similar symmetrical kernel functions. In this situation one no longer speaks of an MLP network, but of an RBF network.

Figure 19.7 shows an incomplete neural network with two output variables. The weights $w_{13}, w_{22}, w_{31}, v_{12}$ and $v_{31}$ are set to 0, and the corresponding edges are not displayed in the network graphs. The output variable $y_1$ is, for example

$$y_1 = v_{11}\psi(w_{01} + w_{11}x_1 + w_{21}x_2) + v_{21}\psi(w_{02} + w_{12}x_1 + w_{32}x_3),$$

a linear combination of the results of the two upper neurons of the hidden layers.

Until now we have only discussed those cases that are most often handled

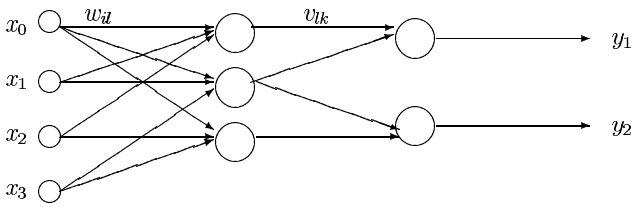


Figure 19.7: Multiple layer perceptron with two hidden layers

in the literature, where a neuron has an effect on the linear combination of variables from the previous layer. Occasionally the case where the output of a neural of the form $\psi\left(\Pi_{i=1}^{p} w_i x_i\right)$ respectively $\psi\left(\max_{i=1,\ldots,p} x_i\right)$ is considered.

Neural networks of MLP types can be used for classification problems as well as for regression and forecast problems. In order to find an adequate network for each problem, the weights have to be learned through a training set, i.e., the network parameters are estimated from the data. Since we are restricting ourselves to the case of supervised learning, this means that $(x^{(1)}, z^{(1)}), \ldots, (x^{(T)}, z^{(T)})$ are given for the training set. The $x^{(i)} \in \mathbb{R}^p$ are input vectors, the $z^{(i)} \in \mathbb{R}^q$ are the corresponding desired output values from the network. The vectors $z^{(i)}$ are compared to the actual output vectors $y^{(i)} \in \mathbb{R}^q$ of the network. The weights are determined so that the deviations between $z^{(i)}$ and $y^{(i)}$ are small. An example of this is the *least squares (LS) application* already mentioned in the analysis of the perceptron:

Assuming that the training set $(x^{(1)}, z^{(1)}), \ldots, (x^{(T)}, z^{(T)})$ is given. The weights $w_{0l}, l = 1, \ldots, r$, $x_0 \stackrel{\text{def}}{=} 1$ are given, where $r$ is the number of neurons in the first hidden layer. The weights of all the other edges in the network (between the input layer, the hidden layers and the output layer) are determined so that

$$\sum_{k=1}^{T} ||z^{(k)} - y^{(k)}||^2 = \min!$$

In the network given in Figure 19.7 the minimisation is carried out with respect to the weights $w_{11}, w_{12}$, $w_{21}, w_{23}, w_{32}, w_{33}, v_{11}, v_{21}, v_{22}, v_{32}$. As for the perceptron the weights $w_{01}$, $w_{02}$, $w_{03}$ can be set in order to avoid the arbitrary choice of scale factors.

Instead of the LS method, other loss functions can also be minimised, for

example, weighted quadratic distances or, above all in classification, the Kullback-Leibler distance:

$$\sum_{k=1}^{T} \sum_{i} \left\{ z_i^{(k)} \log \frac{z_i^{(k)}}{y_i^{(k)}} + (1 - z_i^{(k)}) \log \frac{1 - z_i^{(k)}}{1 - y_i^{k)}} \right\} = \min!$$

Since only the $y_i^{(k)}$ depend on the weights, it is equivalent to minimise the cross-entropie between $z_i$ and $y_i$, which are both contained in $(0, 1)$:

$$-\sum_{k=1}^{T} \sum_{i} \left\{ z_i^{(k)} \log y_i^{(k)} + (1 - z_i^{(k)}) \log(1 - y_i^{(k)}) \right\} = \min!$$

## 19.2 Back Propagation

The most well known method with which the feed forward network learns its weights from the training set is the back propagation. The basic idea is non other than a numerical method to solve the (nonlinear) least squares problem that saves on memory, at the cost however, of possible slower convergence and numerical instabilities.

To illustrate, consider a neural network with an output variable $y$ (i.e. $q = 1$) and a hidden layer with only one neuron:

$$y = \psi(w_0 + w^\top x).$$

$\psi$ can be a logistic function, or some other transformation function. The training set is $(x^{(1)}, z^{(1)}), \ldots, (x^{(T)}, z^{(T)})$. The weight $w_0$ is held constant in order to avoid the arbitrary scale factor. The function to be minimised

$$Q(w) = \sum_{k=1}^{T} (z^{(k)} - y^{(k)})^2$$

is thus only dependent on the weights $w_1, \ldots, w_p$ of the input variables.

An elementary numerical method for minimizing $Q$ is the *decreasing gradient method*. Given a weight $w(N)$ one calculates the next approximation by moving a small step in the direction of the steepest decline of $Q$:

$$\begin{aligned} w(N+1) &= w(N) - \eta \,\mathrm{grad}\, Q(w(N)), \\ \mathrm{grad}\, Q(w) &= -\sum_{k=1}^{T} 2(z^{(k)} - y^{(k)}) \psi'(w^\top x^{(k)}) x^{(k)}. \end{aligned}$$

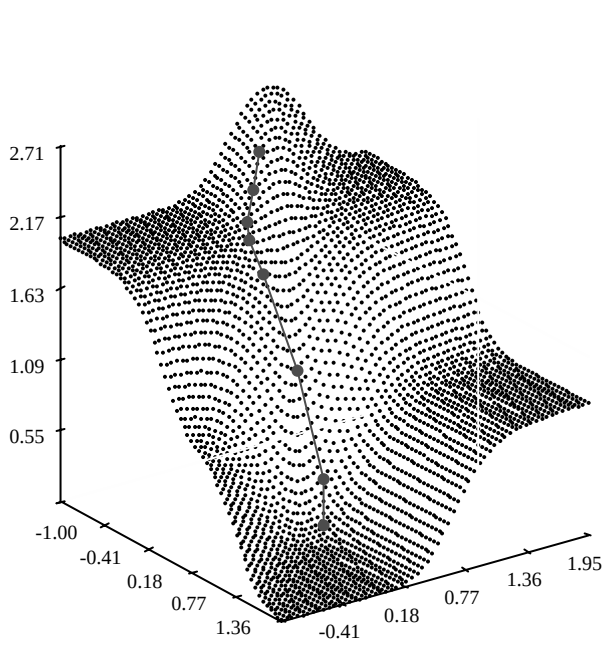


Figure 19.8: Gradients descent proceedings with goal vector $z = (0, 1, 0, 1)^\top$.
SFEdescgrad

To accelerate the convergence, the small constant $\eta > 0$ can also converge to 0 during the iteration process. Figure 19.8 shows the path of optimising $Q(w)$ evaluated in $i$-steps at $w_1, ..., w_i$, where each $w$ is corrected according to the back propagation rule.

With the decreasing gradient method the quality of the weight $w(N)$, that is the actual network, is evaluated simultaneously using all the data in the training set. The network is applied to all $x^{(1)}, \ldots, x^{(T)}$, and only after this the weight vector changed.

*Back propagation* is also a form of the decreasing gradient method with the difference that the network is repeatedly applied to the single $x^{(k)}$ and after

every single step the weights are already changed in the direction of the steepest decline of the function $Q_k(w) = (z^{(k)} - y^{(k)})^2$ :

$$\begin{aligned} w(N+1) &= w(N) - \eta \operatorname{grad} Q_k(w(N)) \ , \ k = 1, \ldots, T, \\ \operatorname{grad} Q_k(w) &= -2(z^{(k)} - y^{(k)})\psi'(w^\top x^{(k)})x^{(k)}. \end{aligned}$$

If in this process the training set has been gone through once, the iteration starts again from the beginning. $T$ steps in the back propagation correspond then roughly to one step in the decreasing gradient method. Also by the back propagation algorithm it may be necessary to allow $\eta$ to converge slowly to 0.

The Widrow-Hoff learning rule is in principle a back propagation algorithm. The threshold function $\psi(t) = \mathbf{1}(w_0 + t > 0)$ is non-differentiable, but after the presentation of $x^{(k)}$ the weights are changed in the direction of the steepest decline of $Q_k(w)$, i.e., in the direction of $x^{(k)}$ for $z^{(k)} = 1, y^{(k)} = 0$ and in the direction of $-x^{(k)}$ for $z^{(k)} = 0,\ y^{(k)} = 1$. By correct classifications the weights here remain unaltered.

Naturally one can apply every numerical algorithm that can calculate the minimum of a non-linear function $Q(w)$ to determine the weights of a neural network. By some applications, for example, the conjugate gradient method has proven to be the fastest and most reliable method. All of these algorithms have the danger of landing in a local minimum of $Q(w)$. In the literature on neural networks it is occasionally claimed that with the combination of training, the networks, such as local minima do not occur. Based on experience of statistics with maximum likelihood estimators of large dimensional parameters, this is to be expected since the training of neural networks for applications of regression analysis, for example, can be interpreted under the appropriate normality assumptions as the maximum likelihood estimation technique.

## 19.3 Neural Networks in Non-parametric Regression Analysis

Neural networks of type MLP describe a mapping of the input variables $x \in \mathbb{R}^p$ onto the output variables $y \in \mathbb{R}^q$. We will restrict ourselves in this section to the case where the network has only one hidden layer and the output variable is univariate ($q = 1$). Then $y \in \mathbb{R}$ as a function of $x$ has the

form

$$y = v_0 + \sum_{h=1}^{H} v_h \psi(w_{0h} + \sum_{j=1}^{p} w_{jh} x_j) \stackrel{\text{def}}{=} \nu_H(x; \vartheta) \tag{19.1}$$

where $H$ is the number of neurons in the hidden layer and $\psi$ is the given transformation function. The parameter vector

$$\vartheta = (w_{01}, \ldots, w_{p1}, w_{02}, \ldots, w_{pH}, v_0, \ldots, v_H)^\top \in \mathbb{R}^{(p+1)H+H+1}$$

contains all the weights of the network. This network with one hidden layer already has a universal approximation property: every measurable function $m : \mathbb{R}^p \to \mathbb{R}$ can be approximated as accurately as one wishes by the function $\nu_H(x, \vartheta)$ when $\psi$ is a monotone increasing function with a bounded range. More precisely, the following result holds, Hornik, Stinchcombe and White (1989):

**Theorem 19.1**
*Let* $\psi : \mathbb{R} \to [0, 1]$ *be monotone increasing with* $\lim_{u \to -\infty} \psi(u) = 0$, $\lim_{u \to \infty} \psi(u) = 1$, *and let* $J = \{\nu_H(x; \vartheta);\ H \geq 1, \vartheta \in \mathbb{R}^{(p+1)H+H+1}\}$ *be the set which is mapped by a MLP function with a hidden layer from* $\mathbb{R}^p$ *to* $\mathbb{R}$.

*a) For every Borel measurable function* $f : \mathbb{R}^p \to \mathbb{R}$ *there a series* $\nu_n \in J$, $n \geq 1$ *exists, with* $\mu(x; |f(x) - \nu_n(x)| > \varepsilon) \longrightarrow 0$ *for* $n \to \infty$, $\varepsilon > 0$, *where* $\mu$ *is an arbitrary probability measure of the Borel-*$\sigma$*-Algebra from* $\mathbb{R}^p$.

*b) For every increasing function* $f : \mathbb{R}^p \to \mathbb{R}$ *there a series* $\nu_n \in J$, $n \geq 1$ *exists, with* $\sup_{x \in C} |f(x) - \nu_n(x)| \longrightarrow 0$ *for* $n \to \infty$, *where* $C$ *is an arbitrary compact subset of* $\mathbb{R}^p$.

The range of $\psi$ can be set to any bounded interval, not only $[0, 1]$, without changing the validity of the approximation properties.

The weight vector $\vartheta$ is not uniquely determined by the network function $\nu_H$. If, for example, the transformation function is asymmetric around 0, i.e., $\psi(-u) = -\psi(u)$, then $\nu_H(x; \vartheta)$ does not change when
a) the neurons of the hidden layer are interchanged, which corresponds to a substitution of the coordinates of $\vartheta$, or when
b) all input weights $w_{0h}, \ldots, w_{ph}$ and the output weight $v_h$ of the neural are multiplied by $-1$.

In order to avoid this ambiguity we will restrict the parameter set to a fundamental set in the sense of Rüeger and Ossen (1997), which for every network

function $\nu_H(x;\vartheta)$ contains exactly one corresponding parameter vector $\vartheta$. In the case of asymmetric transformation functions we restrict ourselves, for example, to weight vectors with $v_1 \geq v_2 \geq \ldots \geq v_H \geq 0$. In order to simplify the following considerations we also assume that $\vartheta$ is contained in a sufficiently large compact subset $\Theta_H \subset \mathbb{R}^{(p+1)H+H+1}$ of a fundamental range.

Due to their universal approximation properties neural networks are a suitable tool in constructing non-parametric estimators for regression functions. For this we consider the following heteroscedastic regression model:

$$Z_t = f(X_t) + \varepsilon_t \ , \quad t = 1, \ldots, n,$$

where $X_1, \ldots, X_n$ are independent, identically distributed $d$-variate random variables with a density of $p(x), x \in \mathbb{R}^d$. The residuals $\varepsilon_1, \ldots, \varepsilon_n$ are independent, real valued random variables with

$$\mathsf{E}(\varepsilon_t | X_t = x) = 0, \quad \mathsf{E}(\varepsilon_t^2 | X_t = x) = s_\varepsilon^2(x) < \infty.$$

We assume that the conditional mean $f(x)$ and the conditional variance $s_\varepsilon^2(x)$ of $Z_t$ are, given $X_t = x$, continuous functions bounded to $\mathbb{R}^d$. In order to estimate the regression function $f$, we fit a neural network with a hidden layer and a sufficiently large number, $H$, of neurons to the input variables $X_1, \ldots, X_n$ and the values $Z_1, \ldots, Z_n$, i.e., for given $H$ we determine the non-linear least squares estimator $\hat{\vartheta}_n = \operatorname{argmin}_{\vartheta \in \Theta_H} D_n(\vartheta)$ with

$$\hat{D}_n(\vartheta) = \frac{1}{n} \sum_{t=1}^{n} \{Z_t - \nu_H(X_t;\vartheta)\}^2 .$$

Under appropriate conditions $\hat{\vartheta}_n$ converges in probability for $n \to \infty$ and a constant $H$ to the parameter vector $\vartheta_0 \in \Theta_H$, which corresponds to the best approximation of $f(x)$ by a function of type $\nu_H(x;\vartheta), \vartheta \in \Theta_H$:

$$\vartheta_0 = \operatorname*{argmin}_{\vartheta \in \Theta_H} D(\vartheta) \quad \text{with} \quad D(\vartheta) = \mathsf{E}\{f(X_t) - \nu_H(X_t;\vartheta)\}^2.$$

Under somewhat stronger assumptions the asymptotic normality of $\hat{\vartheta}_n$ and thus of the estimator $\hat{f}_H(x) = \nu_H(x;\hat{\vartheta}_n)$ also follows for the regression function $f(x)$.

The estimation error $\hat{\vartheta}_n - \vartheta_0$ can be divided into two asymptotically independent sub-components: $\hat{\vartheta}_n - \vartheta_0 = (\hat{\vartheta}_n - \vartheta_n) + (\vartheta_n - \vartheta_0)$, where the value

$$\vartheta_n = \operatorname*{argmin}_{\vartheta \in \Theta_H} \frac{1}{n} \sum_{t=1}^{n} \{f(X_t) - \nu_H(X_t;\vartheta)\}^2$$

minimises the sample version of $D(\vartheta)$, Franke and Neumann (2000):

**Theorem 19.2**
*Let $\psi$ be bounded and twice differentiable with a bounded derivative. Suppose that $D(\vartheta)$ has a unique global minimum $\vartheta_0$ in the interior of $\Theta_H$, and the Hesse matrix $\nabla^2 D(\vartheta_0)$ of $D$ at $\vartheta_0$ is positive definite. In addition to the above mentioned conditions for the regression model it holds that*

$$\begin{array}{rcll} 0 < \delta \leq s_\varepsilon^2(x) & \leq & \Delta < \infty & \text{for all } x, \\ \mathsf{E}(|\varepsilon_t|^\gamma | X_t = x) & \leq & C_\gamma < \infty & \text{for all } x,\ \gamma \geq 1 \end{array}$$

*with suitable constants $\delta, \Delta, C_n, \gamma \geq 1$. Then it holds for $n \to \infty$ :*

$$\sqrt{n} \begin{pmatrix} \hat{\vartheta}_n - \vartheta_n \\ \vartheta_n - \vartheta_0 \end{pmatrix} \xrightarrow{\mathcal{L}} N\left(0, \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}\right)$$

*with covariance matrices*

$$\begin{aligned} \Sigma_i &= \{\nabla^2 D(\vartheta_0)\}^{-1} B_i(\vartheta_0) \{\nabla^2 D(\vartheta_0)\}^{-1}, i = 1, 2, \\ B_1(\vartheta) &= 4 \int s_\varepsilon^2(x)\ \nabla \nu_H(x;\vartheta) \nabla \nu_H(x;\vartheta)^\top\ p(x) dx \\ B_2(\vartheta) &= 4 \int \{f(x) - \nu_H(x;\vartheta)\}^2 \nabla \nu_H(x;\vartheta) \nabla \nu_H(x;\vartheta)^\top p(x) dx \end{aligned}$$

*where $\nabla \nu_H$ represents the gradient of the network function with respect to the parameter $\vartheta$.*

From the theorem it immediately follows that $\sqrt{n}(\hat{\vartheta}_n - \vartheta_0)$ is asymptotically $\mathrm{N}(0,\ \Sigma_1 + \Sigma_2)$ distributed. $\Sigma_1$ here stands for the variability of the estimator $\hat{\vartheta}_n$ caused by the observational error $\varepsilon_t$. $\Sigma_2$ represents the proportion of asymptotic variability that is caused by the mis-specification of the regression function, i.e., from the fact that $f(x)$ is of the form $\nu_H(x;\vartheta)$ for a given $H$ and no $\vartheta$. In the case that it is correctly specified, where $f(x) = \nu_H(x;\vartheta_0)$, this covariance component disappears, since $B_2(\vartheta_0) = 0$ and $\Sigma_2 = 0$.

$\Sigma_1, \Sigma_2$ can be estimated as usual with the sample covariance matrices. In order to construct tests and confidence intervals for $f(x)$ a couple of alternatives to the asymptotic distribution are available: Bootstrap, or in the case of heteroscedasticity, the Wild Bootstrap method, Franke and Neumann (2000).

Theorem 19.2 is based on the theoretical value of the least squares estimator $\hat{\vartheta}_n$, which in practice must be numerically determined. Let $\tilde{\vartheta}_n$ be such a numerical approximation of $\hat{\vartheta}_n$. The quality of the resulting estimator $\tilde{\vartheta}_n$ can depend on the numerical method used. White (1989b) showed in particular that the back propagation algorithm leads, under certain assumptions,

to an asymptotically inefficient estimator $\tilde{\vartheta}_n$, i.e., the asymptotic covariance matrix of $\sqrt{n}(\tilde{\vartheta}_n - \vartheta_0)$ is larger than that of $\sqrt{n}(\hat{\vartheta}_n - \vartheta_0)$ in the sense that the difference of the two matrices is positive definite. Nevertheless White also showed that by joining a single global minimisation step, the estimator calculated from the back propagation can be modified so that for $n \to \infty$ it is as efficient as the theoretical least squares estimator $\hat{\vartheta}_n$.

Until now we have held the number of neurons $H$ in the hidden layer of the network and thus the dimension of the parameter vector $\vartheta$ constant. The estimator based on the network, $\hat{f}_H(x) = \nu_H(x; \hat{\vartheta}_n)$ converges to $\nu_H(x; \vartheta_0)$, so that in general the bias $\mathsf{E}\{\hat{f}_H(x)\} - m(x)$ for $n \to \infty$ does not disappear, but rather converges to $\nu_H(x; \vartheta_0) - f(x)$. With standard arguments it directly follows from Theorem 19.2 that:

**Corollary 19.1** *Under the assumptions from Theorem 19.2 it holds for* $n \to \infty$ *that*

$$\sqrt{n}\left\{\nu_H(x; \hat{\vartheta}_n) - \nu_H(x; \vartheta_0)\right\} \xrightarrow{\mathcal{L}} N(0, \sigma_\infty^2)$$

*with* $\sigma_\infty^2 = \nabla\nu_H(x; \vartheta_0)^\top (\Sigma_1 + \Sigma_2)\, \nabla\nu_H(x; \vartheta_0)$.

In order to obtain a consistent estimator for $f(x)$, the number of neurons $H$, which by the non-parametric estimator $\hat{f}_H(x)$ play the role of a smoothing parameter, must increase with $n$. Due to the universal approximation properties of the neural network $\nu_H(x; \vartheta_0)$ thus converges to $f(x)$, so that the bias disappears asymptotically. Since with an increasing $H$ the dimension of the parameter vector $\vartheta$ increases, $H$ should not approach $\infty$ too quickly, in order to ensure that the variance of $\hat{f}_H(x)$ continues to converge to 0. In choosing $H$ in practice one uses a typical dilemma for non-parametric statistics, the bias variance dilemma: a small $H$ results in a smooth estimation function $\hat{f}_H$ with smaller variance and larger bias, whereas a large $H$ leads to a smaller bias but a larger variability of a then less smoothing estimator $\hat{f}_H$.

White (1990) showed in a corresponding framework that the regression estimator $\hat{f}_H(x)$ based on the neural network converges in probability to $f(x)$ and thus is consistent when $n \to \infty$, $H \to \infty$ at a slower rate.

From this it follows that neural networks with a free choice of $H$ neurons in the hidden layer provides useful non-parametric function estimators in regression, and as we will discuss in the next section, in time series analysis. They have the advantage that the approximating function $\nu_H(x; \vartheta)$ of the

form (19.1) is a combination of the neurons, which are composed of only a given non-linear transformation of an affine-linear combination of the variables $x = (x_1, \ldots, x_d)^\top$. This makes the numerical calculation of the least squares estimator for $\vartheta$ possible even when the dimension $d$ of the input variables and the number $H$ of neurons are large and thus the dimension $(d+1)H + H + 1$ of the parameter vector is very large. In contrast to the local smoothing technique introduced in Chapter 14, the neural networks can also be applied as estimators of functions in large dimensional spaces. One reason for this is the non-locality of the function estimator $\hat{f}_H(x)$. This estimator does not dependent only on the observations $(X_t, Z_t)$ with a small norm $||X_t - x||$ and thus in practice it is not as strongly afflicted by the imprecation of dimensionality, i.e., even for large $n$ there is a smaller local density of the observation $X_t$ in large dimensional spaces.

Theoretically it is sufficient to consider neural networks of type MLP with one hidden layer. In practice, however, one can sometimes achieve a comparably good fit to the data with a relatively more parsimonious parameterization by creating multiple hidden layers. A network function with two hidden layers made up of $H$ and $G$ neurons respectively has, for example, the following form

$$\nu(x;\vartheta) = v_0 + \sum_{g=1}^{G} v_g \psi \left( w'_{0g} + \sum_{h=1}^{H} w'_{hg} \psi(w_{0h} + \sum_{j=1}^{d} w_{jh}\ x_j) \right),$$

where $\vartheta$ represents the vector of all the weights $v_g, w'_{hg},\ w_{jh}$. Such a function with small $H, G$ can produce a more parsimonious parameterized approximation of the regression function $f(x)$ than a network function with only one hidden layer made up of a large number of neurons. In a case study on the development of trading strategies for currency portfolios Franke and Klein (1999) discovered, that with two hidden layers a significantly better result can be achieved than with only one layer.

In addition the number of parameters to be estimated can be further reduced when several connections in the neural network are cut, i.e., when the corresponding weights are set to zero from the very beginning. The large flexibility that the neural network offers when approximating regression functions creates problems when creating the model, since one has to decide on a network structure and thus ask:

1. How many hidden layers does the network have?
2. How many neurons does each hidden layer have?
3. Which nodes (inputs, hidden neurons, outputs) of the network should

be connect, i.e., which weights should be set to zero from the very beginning?

Through this process one is looking for a network which makes it possible to have a network function $\nu(x;\vartheta)$ that is parsimoniously parameterized and at the same time for a suitable $\vartheta$ that is a sufficiently good approximation of the regression function $f(x)$.

Similar to the classical linear regression analysis there are a comprehensive number of instruments available for specifying a network structure consistent with the data. For simplicity we will concentrate on the feed forward network with only one hidden layer made up of $H$ neurons.

a) *Repeated Significance Tests:* As with the stepwise construction of a linear regression model we start with a simple network assuming that one additional neuron with the number $H$ and $v_H$ output weights has been added. Whether in doing this the quality of the fit of the network has significantly improved is determined by testing the hypothesis $H_0 : v_H = 0$ against the alternative $H_1 : v_H \neq 0$. Since under $H_0$ the input weights $w_{0H}, ..., w_{pH}$ of the neurons in question are not identifiable, i.e., they have no influence on the value of the network function $\nu_H$, this is not a standard testing problem. White (1989a), Teräsvirta, Lin and Granger (1993) have developed Lagrange multiplier tests that are suitable for testing the significance of an additional neuron. Going in the other direction it is also possible to start with a complex network with large $H$ assumed neurons and successively removing them until the related test rejects the hypothesis $H_0 : v_H = 0$. To reduce the number of parameters it makes sense to cut individual input connections, i.e., to set the corresponding weight to zero. For the test of the hypothesis $H_0 : w_{jh} = 0$ against the alternative $H_1 : w_{jh} \neq 0$ classical Wald Tests can be applied due to the asymptotical results such as 19.2 (see for example Anders (1997) for applications in financial statistics).

b) *Cross Validation and Validation:* The resulting cross validation is usually eliminated due to the extreme amount of calculations to determine the order of the model, i.e., first of all the number $H$ of neurons in the hidden layer. In order to calculate the leave-one-out estimator for the model parameters one must fit the neural network to the corresponding sample that has been reduced by one observation a total of $n$ times, and this must be done for every network structure under consideration. A related and more known procedure from the application of neural networks in the regression and time series analysis is to take a portion of the data away from the sample in order to measure the quality of the model based on this so called validation set. In

addition to the data $(X_t, Z_t),\ t = 1, \ldots, n$, used to calculate the least squares estimator $\hat{\vartheta}_n$ a second independent subsample $(X_t, Z_t),\ t = n+1, \ldots, n+M$, is available. By minimising measurements of fit, such as,

$$V(H) = \frac{1}{M} \sum_{t=n+1}^{n+M} \left\{ Z_t - \nu_H(X_t; \hat{\vartheta}_n) \right\}^2$$

the order of the model $H$ and the quality of the incomplete network structure can be determined, in which individual input weights have been set to zero.

c) *Network Information Criteria:* To compare the network structures some well known applications for determining order, such as the Akaike Information Criterion (AIC), can be used. The application from Murata, Yoskizawa and Amari (1994) called the *Network Information Criterion* (NIC) is specialized for the case of neural networks. Here it is implicitly assumed that the residuals $\varepsilon_t$ are normally distributed with a common variance $\sigma_\varepsilon^2$.

## 19.4 Forecasts of Financial Time Series with Neural Networks

To forecast the future development of financial time series an autoregressive model is particularly suitable. The value of the time series at date $t + 1$ is a function of many infinite observations from the past, in addition to an innovation independent of the past:

$$Z_{t+1} = f(Z_t, \ldots, Z_{t-p+1}) + \varepsilon_{t+1}, \quad -\infty < t < \infty, \tag{19.2}$$

where $\varepsilon_t,\ -\infty < t < \infty$, is independently and identically distributed with $\mathsf{E}(\varepsilon_t) = 0,\ Var(\varepsilon_t) = \sigma_\varepsilon^2 < \infty$. The analogy of this formula for this nonlinear autoregressive model of order $p$ (NLAR($p$)) to the regression model considered in a previous section is obvious, where the $p$-variate random vector $(Z_t, \ldots, Z_{t-p+1})^\top$ takes the place of the $d$-variate independent variable $X_t$. The autoregression function $f : \mathbb{R}^p \to \mathbb{R}$ in this model immediately gives the best forecast for $Z_{t+1}$ given the value of the time series up to date $t$:

$$\hat{Z}^0_{t+1|t} = f(Z_t, \ldots, Z_{t-p+1}).$$

Since $f$ is in general not known, it seems obvious in view of the last section to approximate the autoregression function with a neural network when observations of the times series $Z_1, \ldots, Z_{n+1}$ are available. For training the network,

i.e., for estimating the network weights, the vector $(Z_t, \ldots, Z_{t-p+1})^\top$ is used as input values and as output values $Z_{t+1}$ for $t = p, \ldots, n$, is used. We will restrict ourselves for simplicity to the MLP with one hidden layer. $\hat{\vartheta}_n$ again represents the least squares estimator for the weight vector:

$$\hat{\vartheta}_n = \underset{\vartheta \in \Theta_H}{\operatorname{argmin}} \frac{1}{n-p+1} \sum_{t=p}^{n} \{Z_{t+1} - \nu_H\,(Z_t, \ldots, Z_{t-p+1}; \vartheta)\}^2$$

where $\nu_H$ is defined as in the previous section. We thus obtain a non-parametric forecast based on a neural network for $Z_{t+1}$ :

$$\hat{Z}_{t+1|t} = \nu_H(Z_t, \ldots, Z_{t-p+1}; \hat{\vartheta}_n).$$

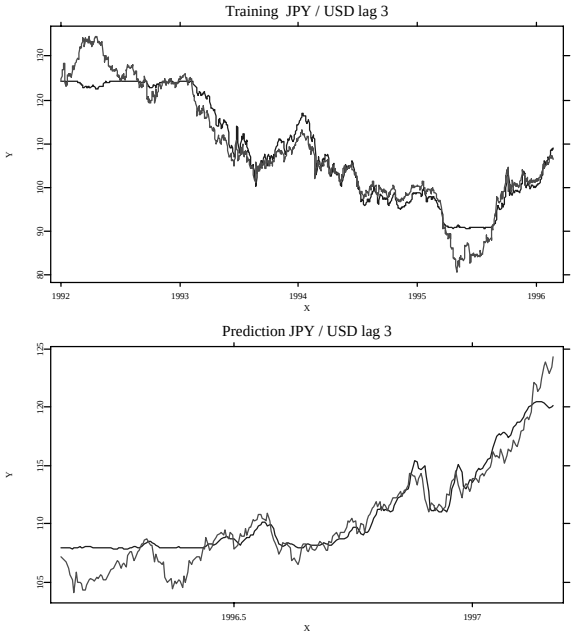


Figure 19.9: Approximation of exchange rate JPY/USD (red) through RBF neural network (blue): Training set(above) and forecasts(below)
SFEnnjpyusd

The result of this procedure is illustrated in Figure 19.9: it shows the forecasting of the exchange rate time series JPY/USD using neural networks

considering 3 periods of time dependency.

The asymptotic normality of the parameters and of the function estimators and the consistency of $\nu_H(\cdot;\hat{\vartheta}_n)$ as an estimator of $f$ for an increasing $H$ remain robust even in the event where the stochastic process $\{Z_t,\ -\infty < t < \infty\}$ is $\alpha$-mixing with exponentially decreasing mixing coefficients, White (1989b) and White (1990). Franke, Kreiss, Mammen and Neumann (2003) have formulated conditions for the case where $p = 1$ for the autoregression function $f$ and for the distribution of the innovations $\varepsilon_t$, which guarantee for the NLAR(1) process the strongest $\beta$-mix properties with exponentially decreasing coefficients. Next to technical details it is essential that

$$\lim_{|x|\to\infty} |f(x)/x| < 1$$

is fulfilled, because it is sufficient for the innovation distribution that the density does not vanish anywhere. The last condition can be considerably weakened.

The conditions on the autoregression function is comparatively weak and obvious when one considers the stationarity conditions $|\alpha| < 1$ for linear AR(1) processes $Z_{t+1} = \alpha\, Z_t + \varepsilon_{t+1}$, where $f(x) = \alpha\, x$. Accordingly also for NLAR($p$) process of large order ($p > 1$) it is sufficient to use weaker conditions on $f$, which above all guarantees stationarity in order to make the neural network a useful tool as a non-parametric estimator of $f$.

For the practical forecast one not only wants to use the last values in the time series, but also economic data available at time $t$ such as exchange rates, index values, oil prices or the non-linear transformation of prices. To do this the non-linear autoregressive process with exogenous components of order $p$ (NLARX($p$)) process is suitable:

$$Z_{t+1} = f(Z_t, \ldots, Z_{t-p+1}, X_t) + \varepsilon_t,\ \ -\infty < t < \infty, \tag{19.3}$$

where the innovations $\varepsilon_t,\ -\infty < t < \infty$, are again independently and identically distributed with $\mathsf{E}(\varepsilon_t) = 0,\ \ Var(\varepsilon_t) = \sigma_\varepsilon^2 < \infty$, and $X_t$ is the value of a $d$-variate stochastic process that contains all external information available at date $t$, which is used in the forecast.

The practical application of the forecast on financial time series with neural networks is illustrated with a pilot study that was carried out in cooperation with Commerzbank AG, Franke (1999). The goal was to develop a trading strategy for a portfolio made up of 28 of the most important stocks from the Dutch CBS-Index. We will restrict ourselves here to the buy-and-hold strategy with a time horizon of a quarter of a year (60 trading days), i.e., the portfolio is created at the beginning of a quarter and then held for three

months with no alterations. At the end of the three months the value of the portfolio should be as large as possible.

As a basis for the trading strategy a three month forecast of the stocks is used. $S_t$ represents the price of one of the 28 stocks. To model the time series $S_t$ we use a NLARX process of the form (19.3); the system function $f$ is approximated with a network function $\nu_H(S_t, A_t, X_t; \vartheta)$. Here $A_t$ is a vector made up of constant non-linear transformations of $S_t, ..., S_{t-p+1}$ that were taken from the technical market analysis, for example, a moving average, momentum or Bollinger-intervals, Müller and Nietzer (1993), Welcker (1994). The random vector $X_t$ represents the chosen market data such as index prices, exchange rates, international interest rates, etc. As is expected with a forecast horizon of 60 units of time into the future, the actual forecasts of the stock prices in 60 days,

$$\hat{S}_{t+60|t} = \nu_H(S_t, A_t, X_t; \hat{\vartheta}_n),$$

is not very reliable. For making the decision as to whether a stock should be included in the portfolio or not, the general trend of the price developments are most important instead of the actual price of the stock at the end of the holding period. To realise this aspect in formulating the portfolio, it should be considered whether based on the network based forecast, $\hat{S}_{t+60|t}$, the price is expected to increase considerably (more than 5 %), decrease considerably (more than 5 %) or whether it is essentially expected to stay at the same level. The network based portfolio is composed of those stocks (with relative proportions that are taken from the stock's corresponding weight in the CBS Index) for which $(\hat{S}_{t+60|t} - S_t)/S_t > 0.05$. Here the *same* network function $\nu_H(S_t, A_t, X_t; \vartheta)$ is used for all 28 stocks taken into consideration whose price dependent arguments $S_t$ actually take on the stock specific values.

In choosing a suitable network and in estimating the network weight vector $\vartheta$ the data from 1993 to 1995 is used. In choosing the network structure a statistical model selection technique and the experience of experts was used. The resulting network is a multiple layered perceptron with one hidden layer made up of $H = 3$ neurons. The input vector $(S_t, A_t, X_t)$ has the dimension 25, so that a parameter vector $\vartheta \in \mathbb{R}^{82}$ needed to be estimated.

To check the quality of the network based trading strategy, it is applied to the data from 1996. At the beginning of every quarter a portfolio made up of 28 stocks is created based on the network based forecast. At the end of the quarter the percentage increase in value is considered. As a comparison the increase in value of a portfolio exactly replicating the CBS Index is considered. Since in the years considered the market was of the most part in an increasing phase, it is known from experience that it is hard to beat an index. As Table 19.1 shows, the network portfolio achieved a higher per-

centage increase in value in every quarter than the index portfolio, that is in the quarters, such as the first and fourth, where the index had substantially increased, as well as in the quarters, such as the second, where the index had minimally decreased. Nevertheless the results need to be interpreted with a bit of caution. Even in the training phase (1993-1995) the CBS Index tended to increase, so that the network was able to specialise in a trend forecast in a generally increasing market. Presumably one would need to use a different network as a basis for the trading strategy, when the market fluctuates within a long-term lateral motion or when the index dramatically decreases.

| | Quarterly returns | | | |
|---|---|---|---|---|
| | I. | II. | III. | IV. |
| Network portfolio | 0.147 | 0.024 | 0.062 | 0.130 |
| Index portfolio | 0.109 | -0.004 | 0.058 | 0.115 |

Table 19.1: Quarterly returns of a network portfolio and the index portfolio in 1996.

## 19.5 Quantifying Risk with Neural Networks

The previous chapters introduced the most popular measurements of risk, volatility and Value-at-Risk. Both are most often defined as conditional standard deviations or as conditional quantiles respectively, based on a given historical information set. As with other non-parametric methods the neural network can also be used to estimate these measurements of risk. The advantage of the neural network based volatility and VaR estimators lies in the fact that the information used for estimating the risk can be represented by a large dimensional data vector without hurting the practicality of the method. It is possible, for example, to estimate the conditional 5% quantile of the return process of a stock from the DAX given the individual returns of all of the DAX stocks and additional macroeconomic data such as interest rates, exchange rates, etc. In the following section we briefly outline the necessary procedure.

As in (14.1) we assume a model of the form

$$Z_{t+1} = f(Z_t, \ldots, Z_{t-p+1}, X_t) + s(Z_t, \ldots, Z_{t-p+1}, X_t)\,\xi_{t+1} \tag{19.4}$$

to estimate the volatility, where $\xi_t$ are independent, identically distributed random variables with $\mathsf{E}(\xi_t) = 0$, $\mathsf{E}(\xi_t^2) = 1$. $X_t \in \mathbb{R}^d$ represents, as in the previous section, the exogenous information available at date $t$ which we will use in estimating the risk of the time series $Z_t$. The time series given by (19.4) is a non-linear AR(p) ARCH(p) process with exogenous components.

To simplify we use $Z_t(p) = (Z_t, \ldots, Z_{t-p+1})^\top \in \mathbb{R}^p$. It then holds for $z \in \mathbb{R}^p$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} \mathsf{E}[Z_{t+1}|Z_t(p) = z,\ X_t = x] &= f(z,x) \\ Var[Z_{t+1}|Z_t(p) = z, X_t = x] &= s^2(z,x) \\ &= \mathsf{E}[Z_{t+1}^2|Z_t(p) = z, X_t = x] - f^2(z,x). \end{aligned}$$

The conditional expectation function $f(z,x)$ is approximated as in the previous section by a neural network function $\nu_H(z,x;\vartheta)$ of the form (19.1). With the non-linear least squares estimator $\hat{\vartheta}_n$ we obtain for $\vartheta$ an estimator for $f$ :

$$\hat{f}_H(z,x) = \nu_H(z,x;\hat{\vartheta}_n).$$

Analogously we could estimate the conditional mean

$$\mathsf{E}[Z_{t+1}^2|Z_t(p) = z,\ X_t = x] = g(z,x)$$

by approximating the function with a neural network with output function $\nu_G(z,x;\delta)$ and estimate its parameter $\delta$ with a least squares estimator $\hat{\delta}$ within a sufficiently large compact subset $\Delta_G \subset \mathbb{R}^{(p+d+1)G+G+1}$, such as $\Theta_H$, chosen from a fundamental range:

$$\hat{\delta}_n = \underset{\delta\in\Delta_G}{\operatorname{argmin}} \frac{1}{n-p+1} \sum_{t=p}^{n} \left\{ Z_{t+1}^2 - \nu_G(Z_t(p), X_t; \delta) \right\}^2,$$

$$\hat{g}_G(z,x) = \nu_G(z,x;\hat{\delta}_n).$$

As an estimator for the conditional volatility we immediately obtain:

$$\hat{s}_{H,G}^2(z,x) = \hat{g}_G(z,x) - \hat{f}_H^2(z,x).$$

This estimator is in general guaranteed to be positive only for $G = H$. In order to avoid this restriction one can follow the procedure used by Fan and Yao (1998), who have studied a similar problem for the kernel estimator of the conditional variance in a heteroscedastic regression model. Using this application the residuals

$$\varepsilon_{t+1} = Z_{t+1} - f(Z_t(p), X_t) = s(Z_t(p), X_t)\,\xi_{t+1}$$

are approximated by the sample residuals

$$\hat{\varepsilon}_{t+1} = Z_{t+1} - \hat{f}_H(Z_t(p), X_t),\ t = p, \ldots, n,$$

Since the $\xi_{t+1}$ has mean 0 and variance 1,

$$\mathsf{E}[\varepsilon_{t+1}^2 | Z_t(p) = z,\ X_t = x] = s^2(z, x).$$

We could approximate this function directly with a neural network with $G$ neurons and the output function $\nu_G(z, x; \delta)$, whose parameter are estimated by

$$\hat{\delta}_n = \underset{\delta \in \Delta_G}{\operatorname{argmin}} \frac{1}{n - p + 1} \sum_{t=p}^{n} \left\{ \hat{\varepsilon}_{t+1}^2 - \nu_G(Z_t(p), X_t; \delta) \right\}^2.$$

The resulting estimators for the conditional volatility, which through the $\hat{\varepsilon}_t$ is also dependent on $H$, is then

$$\hat{s}_{H,G}(z, x) = \nu_G(z, x; \hat{\delta}_n).$$

Figure 19.10 shows the conditional volatilities estimated from the log returns of the exchange rate time series BP/USD together with some financial indicators using the procedure described above (3 periods are considered as time dependency and radial basis functions networks are used).

It is for arbitrary $G, H$ automatically non-negative. Since the number of neurons essentially determines the smoothness of the network function, it can make sense when approximating $f$ and $s^2$ to choose different networks with $H \neq G$ neurons when it is believed that the smoothness of both functions are quite different from each other.

When the distribution of the innovations $\xi_t$ is additionally specified in the model (19.4), we immediately obtain together with the estimators of $f$ and $s^2$ an estimator of the conditional Value-at-Risk. If the distribution of $\xi_t$ is, for example, $\mathrm{N}(0, 1)$, then the conditional distribution of $Z_{t+1}$ given the information $Z_t(p)$ and $X_t$ at date $t$ is also a normal distribution with mean $f(Z_t(p), X_t)$ and variance $s^2(Z_t(p), X_t)$. If $q_\alpha^\circ$ is the $\alpha$ quantile of the standard normal distribution, then the VaR process $\{Z_t\}$, i.e., the conditional $\alpha$ quantile of $Z_{t+1}$ given $Z_t(p), X_t$ is:

$$VaR_{t+1} = f(Z_t(p), X_t) + s(Z_t(p), X_t) q_\alpha^\circ.$$

An estimator for this conditional Value-at-Risk based on a neural network can be obtained by replacing $f$ and $s$ with the appropriate estimator:

$$\widehat{VaR}_{t+1} = \hat{f}_H(Z_t(p), X_t) + \hat{s}_{H,G}^2(Z_t(p), X_t) q_\alpha^\circ. \tag{19.5}$$

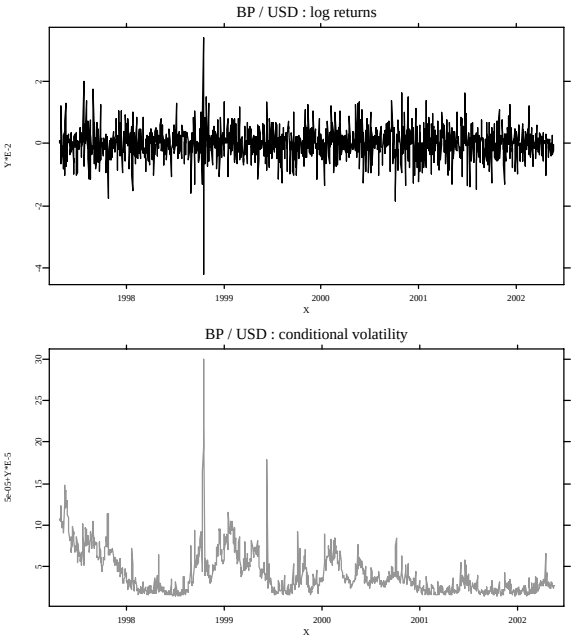


Figure 19.10: Log-returns of exchange rate BP/USD and the estimated conditional variances by RBF neutral network. SFEnnarch

In doing this we can replace the standard normal distribution with another distribution, for example, with a standardised $t$-distribution with mean 0 and variance 1. $q_\alpha^\circ$ is then the corresponding $\alpha$ quantile of the innovation distribution, i.e., the distribution of $\xi_t$.

The estimator (19.5) for the Value-at-Risk assumes that $Z_t$ is a non-linear ARX-ARCHX process of the form (19.4). Above all, however, it has the disadvantage of depending on the critical assumption of a specific distribution of $\xi_t$. Above all the above mentioned procedure, in assuming a stochastic volatility model from the standard normal distribution, has recently been criticised in financial statistics due to certain empirical findings. The thickness of the tails of a distribution of a financial time series appears at times to be so pronounced that in order to adequately model it, even the distribution of the innovations must be assumed to be leptokurtic. Due to the simplicity of the representation a $t$-distribution with only a few degrees of freedom is often considered. In order to avoid the arbitrariness in the choice of the dis-

tribution of the innovations, it is possible to estimate the conditional quantile directly without relying on a model of the form (19.4). This application goes back to the regression quantile from Koenker and Bassett and has been applied by Abberger (1997) to time series in connection with kernel estimation. We assume that $Z_t$ is a stationary time series. As in Chapter 18 $\mathrm{P}_{t+1}$ represents the forecast distribution, i.e., the conditional distribution of $Z_{t+1}$ given $Z_t(p), X_t$. With $F_{t+1}$ we depict the corresponding conditional distribution function

$$F_{t+1}(y|z,x) = \mathrm{P}(Z_{t+1} \leq y | Z_t(p) = z, X_t = x)$$

for $y \in \mathbb{R},\ z \in \mathbb{R}^p, x \in \mathbb{R}^d$. $q_\alpha(z,x)$ is the conditional $\alpha$ quantile, i.e., the solution to the equation $F_{t+1}(q_\alpha(z,x)|z,x) = \alpha$. The conditional quantile function $q_\alpha(z,x)$ solves the minimisation problem

$$\mathsf{E}\{\alpha(Z_{t+1}-q)^+ + (1-\alpha)(Z_{t+1}-q)^- | Z_t(p) = z,\ X_t = x\} = \min_{q\in\mathbb{R}}! \quad (19.6)$$

where $y^+ = y - \mathbf{1}(y \geq 0)$ and $y^- = |y| \cdot \mathbf{1}(y \leq 0)$ represent the positive and negative parts of $y \in \mathbb{R}$. In order to estimate the quantile function directly with a neural network with $H$ neurons, we approximate $q_\alpha(z,x)$ with a network function $\nu_H(z,x;\gamma)$ of the form (19.1), whose weight parameter $\gamma$ lies in a fundamental range $\Gamma_H \subset \mathbb{R}^{(p+d+1)H+H+1}$. $\gamma$ is estimated, however, not with the least squares method, but with the minimisation of the corresponding sample values from (19.6):

$$\hat{\gamma}_n = \operatorname*{argmin}_{\gamma\in\Gamma_H} \frac{1}{n-p+1} \sum_{t=p}^{n} \{\alpha[Z_{t+1} - \nu_H(Z_t(q), X_t)]^+$$

$$+(1-\alpha)[Z_{t+1} - \nu_H(Z_t(q), X_t)]^-\}.$$

As an estimator for the quantile function we obtain

$$\hat{q}_{H\alpha}(z,x) = \nu_H(z,x;\hat{\gamma}_n)$$

and with this the estimator for the conditional Value-at-Risk given $Z_t, \ldots, Z_{t-p+1}, X_t$

$$\widehat{VaR}_{t+1} = \hat{q}_{H\alpha}(Z_t, \ldots, Z_{t-p+1}, X_t).$$

White has shown that under suitable assumptions the function estimators $\hat{q}_{H\alpha}(z,x)$ converge in probability to $q(z,x)$ when the sample observations $n \to \infty$ and when at the same time the number of neurons $H \to \infty$ at a suitable rate.

## 19.6 Recommended Literature

One for mathematicians, statisticians and economists as well, an accessible introduction to the area of the neural network is, e.g., Refenes (1995a). Haykin (1999) offers a comprehensive and effective overview about different forms and applications of neural network. Anders (1997) introduces neural networks from econometrical and statistical view and discusses applications from the finance mathematical areas such as option pricing and insolvency prediction. Ripley (1996) discusses the application of neural network to classification problems in detail and puts them in the context of the classic discriminant analysis. Numerous practical applications of neural network in the finance area are introduced in Rehkugler and Zimmermann (1994), Refenes (1995b), Bol, Nakhaeizadeh and Vollmer (1996) and Franke (2000). The application described in the previous section, calculating the Value-at-Risk by adaptation of a non-linear ARCHX process based on DAX stocks is described in Franke and Diagne (2002).

# 20 Volatility Risk of Option Portfolios

In this chapter we analyse the principal factors in the dynamic structure of implied volatility *at the money* (ATM). The data used are daily *Volatility*-DAX (VDAX) values. By using principal component analysis we consider a method of modelling the risk of option portfolios on the basis of "Maximum Loss".

There is a close connection between the value of an option and the volatility process of the financial underlying. Assuming that the price process follows a geometric Brownian motion we have derived the Black-Scholes formula (BS) for pricing European options in Chapter 6. With this formula the option price is, at a given time point, a function of the volatility parameters when the following values are given: $\tau$ (time to maturity in years), $K$ (strike price), $r$ (risk free, long-run interest rate) and $S$ (the spot price of the financial underlying).

Alternatively one can describe the observed market price of an option at a specific time point with the help of the BS formula using the so called "implied" volatility (see Chapter 6). In doing this one typically finds a U-shaped form for the resulting surface of the volatility over different times to maturity and strike prices. This phenomenon is also referred to as the "Volatility Smile". Figure 20.1 illustrates the typical form of a volatility surface using DAX options. Shown is the implied volatility as a function of the moneyness and the remaining time to maturity $\tau$. Here the term *moneyness* $\frac{S}{K}$ refers to the ratio of the actual price $S$ of the financial underlying and the strike price $K$ of the respective option. It should be noted that options are only traded on the market on a discrete price basis and a discrete time to maturity. In determining the volatility surface, as in Chapter 14, a smoothing technique needs to be applied.

By observing the volatility surface over time, distinct changes in the location and structure become obvious. Identifying the temporal dynamics is of central importance for a number of financially oriented applications. This is of particular importance for the risk management of option portfolios. To

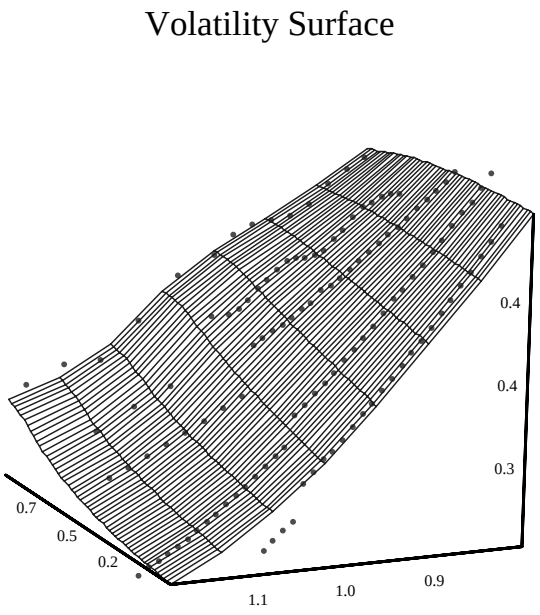


Figure 20.1: Implied volatility surface of the DAX option on 18 July 1998 SFEVolSurfPlot

determine the volatility's dynamics, an application of principal component of analysis is quite suitable. The total temporal structure can be sufficiently represented by a small number of principal components so that the dimensions of the factor space for the purpose of risk analysis can be significantly reduced.

## 20.1 Description of the Data

DAX options belong to the most frequently traded derivatives of the German/Swiss derivative market "EUREX". On every trading day one can find a significant number of liquid time series with varying strike prices and maturities $(K, \tau)$ on the market, which, in principle, can be used to calculate implied volatilities. In view of the often limited data processing capacities, an updated calculation of numerous volatilities and partial derivatives of an

extensive option portfolio is still not feasible. Even with the appropriate available information the isolated consideration of each implied volatility as a separate source of risk is problematic, since it results in an unstructured or "uneven" volatility surface. If one were to use generated volatilities in calibrating option prices, respective risk models, it could lead to serious specification errors and significantly deteriorate the results of the corresponding trading and hedging strategies. As a result of principal component analysis a "smooth" volatility surface, in contrast to the one outlined above, can be generated with a manageable amount of information. This allows for a better calibration of the model and a more precise estimate of portfolio sensitivities.

For our study of the dynamics of implied volatility we use the volatility index (VDAX) made available by the German Stock Exchange (Deutsche Börse AG) respectively the closing prices of the corresponding VDAX subindices. These indices reflect the implied volatility of the DAX options "at the money" for times to maturity from one to 24 months. The corresponding values are determined by applying the Black-Scholes formula (6.23) using prices observed on the market:

$$C(S,\tau) = e^{(b-r)\tau} S\Phi(y + \sigma\sqrt{\tau}) - e^{-r\tau} K\Phi(y),$$

where $\Phi$ is the distribution function of the standard normal distribution and

$$y = \frac{\ln \frac{S}{K} + (b - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}},$$

The only parameter from the BS formula that cannot be immediately observed on the market is the actual volatility $\sigma$ of the price process. In principle the volatility of the process can be estimated from historical financial market data, see Section 6.4.5, however, it is commonly known that the assumption of the BS model, that the financial underlying has a geometric brownian motion, is in reality only approximately fulfilled. Alternatively the BS formula is also used in order to calculate the $\sigma$ value as the implied volatility for a given market price of a specific option. This does not mean that the market participant should accept the assumption of the Black-Scholes method. On the contrary they use the BS formula as a convenient possibility to quote and price options with these parameters.

Given the observed implied volatilities from varying times to maturity $\tau$ at a specific time point and from a strike price $K$, the expectations of the market participants with respect to the future actual volatility of the underlying financial instrument can be estimated. In doing so one must remember that the implied volatility of the BS model does not directly apply to the actual

variance of the price's process. Although the implied BS volatility reflects a market expectation, the theoretical relationship between it and the actual volatility can only be determined using specific assumptions, see Schönbucher (1999), Härdle and Hafner (2000).

Implied volatility for ATM-DAX options are calculated for various lengths of maturity by the German Stock Exchange AG. A detailed description of how the VDAX and its sub-indices are calculated can be found in Redelberger (1994). Since 18 March 1996 maturities of 1, 2, 3, 6, 9, 12, 18 and 24 months have been considered in the calculation. On this date the trading of so called "Long Term Options", i.e., trading of options with maturities of over 12 months, were added to the EUREX. Using closing prices the German Stock Exchange AG calculates a total of eight VDAX sub-indices for the maturities mentioned above for every trading day. These sub-indices reflect the volatility of the respective DAX option "at the money". The time to maturity structure for DAX options that are "at the money" can be determined for every trading day using the VDAX indices. Figure 20.2 illustrates a typical development of the structure, which shows strong changes in the positioning and form of the structure over time.

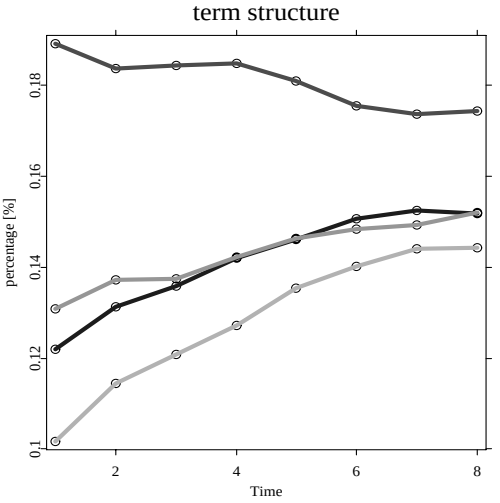


Figure 20.2: Time to maturity structure of implied DAX volatilities "at the money" SFEVolaTermStructure

The analysis carried out here is not only restricted to specific maturities of liquid options, which are represented by the first four VDAX sub-indices. On the contrary, we include all eight sub-indices in the analysis for the following reasons:

First of all a brisk trade in even the "most distant" option contracts (i.e., the contracts with a remaining time of more than one year) take place on numerous trading days, so that excluding the pertaining sub-indices from the analysis would result in a loss of information. VDAX sub-indices for long maturities have been calculated by the German Stock Exchange since 18 March, 1996. After 19 December, 1997 the quality of the data available to us declined considerably. In addition to the daily often unchanged prices, the entries corresponding to the removed sub-indices were usually missing. Given this we have restricted our analysis to the period from 18 March, 1996 to 19 December, 1997.

Including relatively non-liquid DAX options with long maturities appears to make sense for another reason: For our analysis we require constant option maturities, since the daily shortening of the time to maturity can lead to enormous biases in the analysis results with data that has not been corrected. This especially holds for options with a very short time to maturity. Thus we find it utterly necessary to use interpolated volatilities with corresponding constant time to maturities of the underlying option. Referring back to the calculation of the VDAX used by the German Stock Exchange AG we use the following linear interpolation:

For a fixed time to maturity of $\tau_1^* = 30, \tau_2^* = 60, \tau_3^* = 90, \tau_4^* = 180, \tau_5^* = 270, \tau_6^* = 360, \tau_7^* = 540, \tau_8^* = 720$ calendar days we calculate daily volatility indices $\hat{\sigma}_{I,t}(\tau_j^*), j = 1, ..., 8$, using the VDAX sub-indices with the next shorter respectively longer maturity $\hat{\sigma}_{I,t}(\tau_j^-)$ and $\hat{\sigma}_{I,t}(\tau_j^+)$ with

$$\hat{\sigma}_{I,t}(\tau_j^*) = \hat{\sigma}_{I,t}(\tau_j^-) \left[ 1 - \frac{\tau_j^* - \tau_j^-}{\tau_j^+ - \tau_j^-} \right] + \hat{\sigma}_{I,t}(\tau_j^+) \left[ \frac{\tau_j^* - \tau_j^-}{\tau_j^+ - \tau_j^-} \right]. \tag{20.1}$$

This way, we obtain 8 volatility time series each with constant maturities. Every time series represents a weighted average of two consecutive VDAX sub-indices and is based on $n = 441$ daily observations of the implied DAX volatilities "at the money".

| Sub 1 | Sub 2 | Sub 3 | Sub 4 | Sub 5 | Sub 6 | Sub 7 | Sub 8 |
|---|---|---|---|---|---|---|---|
| **20.8** | 9.06 | 6.66 | 6.84 | 4.29 | 2.48 | 2.11 | 1.38 |
| 9.06 | **9.86** | 6.67 | 4.44 | 3.21 | 1.72 | 1.11 | 0.92 |
| 6.66 | 6.67 | **6.43** | 3.87 | 2.63 | 1.49 | 1.01 | 0.53 |
| 6.84 | 4.44 | 3.87 | **4.23** | 2.66 | 1.39 | 1.38 | 0.68 |
| 4.29 | 3.21 | 2.63 | 2.66 | **2.62** | 1.03 | 1.02 | 0.51 |
| 2.48 | 1.72 | 1.49 | 1.39 | 1.03 | **2.19** | 0.63 | 0.33 |
| 2.11 | 1.11 | 1.01 | 1.38 | 1.02 | 0.63 | **1.76** | 0.43 |
| 1.38 | 0.92 | 0.53 | 0.68 | 0.51 | 0.33 | 0.43 | **1.52** |

Table 20.1: Empirical covariance matrix $\hat{\Omega}$ of the first differences (all values have been multiplied by $10^5$) SFEVolaCov

## 20.2 Principal Component Analysis of the VDAX's Dynamics

We will first check the data with the help of the "Augmented Dickey-Fuller" Tests (ADF-Test - see (11.46)) for stationarity. The null hypothesis of a unit root for the individual VDAX sub-indices $\hat{\sigma}_I(\tau_j^*)$ cannot be rejected at the 90% significance level. Obviously due to this result the first differences $x_{jt} = \Delta[\hat{\sigma}_{I,t}(\tau_j^*)] = \hat{\sigma}_{I,t+1}(\tau_j^*) - \hat{\sigma}_{I,t}(\tau_j^*), t = 1, ...., n-1$, of the implied volatility indices will be used for further analysis. Additional ADF tests support the assumption of stationarity for the first differences. SFEAdfKpss

Let $\bar{x}_j$ be the respective sample mean of the first differences $x_{jt}$. Table 20.1 contains the empirical covariance matrix $\hat{\Omega}$ used as an estimator for the $8 \times 8$ matrix $\Omega$ of the covariance $\mathrm{Cov}(x_{it}, x_{jt}), i, j = 1, ..., 8$. With help of the Jordan decomposition we obtain $\hat{\Omega} = \hat{\Gamma}\hat{\Lambda}\hat{\Gamma}^\top$. The diagonal matrix $\hat{\Lambda}$ contains the eigenvalues $\hat{\lambda}_k, k = 1, ..., 8$ of $\hat{\Omega}$, $\hat{\Gamma}$ are the eigenvectors. Time series of the principal components can be obtained with the help of $Y = X_C\hat{\Gamma}$, where $X_C$ represents the $440 \times 8$ matrix of the centered first differences $x_{jt}^c = x_{jt} - \bar{x}_j, j = 1, ..., 8, t = 1, ..., 440$,. The $440 \times 8$ matrix $Y = (Y_1, ..., Y_8), Y_j = (y_{1j}, y_{2j}, ..., y_{440,j})^\top$ contains the principal components.

How accurately the first $l$ principal components have already determined the process of the centered first differences can be measured using the proportion of variance $\varphi_l$ with respect to the total variance of the data. The proportion of explained variance corresponds to the relative proportion of the corresponding eigenvalue, i.e.,

| **Principal Component** | *Explaining proportion of variance* | *cumulative proportion* |
|---|---|---|
| **1** | 70.05 | 70.05 |
| **2** | 13.06 | 83.12 |
| **3** | 5.57 | 88.69 |
| **4** | 3.11 | 91.80 |
| **5** | 3.06 | 94.86 |
| **6** | 2.12 | 96.97 |
| **7** | 1.93 | 98.90 |
| **8** | 1.10 | 100.00 |

Table 20.2: Explained sample variance using principal components in percentage SFEVolaPCA

$$\varphi_l = \frac{\sum_{k=1}^{l} \lambda_k}{\sum_{k=1}^{8} \lambda_k} = \frac{\sum_{k=1}^{l} \mathrm{Var}(y_{tk})}{\sum_{k=1}^{8} \mathrm{Var}(y_{tk})}, \tag{20.2}$$

where $\lambda_k, k = 1, ..., 8$ are the eigenvalues of the true covariance matrix $\Omega$. An estimator for $\varphi_l$ is

$$\hat{\varphi}_l = \frac{\sum_{k=1}^{l} \hat{\lambda}_k}{\sum_{k=1}^{8} \hat{\lambda}_k}.$$

In Table 20.2 the individual proportions of the variance $\hat{\lambda}_l / \sum_{k=1}^{8} \hat{\lambda}_k$ as well as the cumulative variance from the $l$ decomposed proportions from the principal components, $\hat{\varphi}_l$, are displayed. It is obvious that the first principal component already describes 70% of the total variance of the underlying data. With the second principal component an additional 13% of the total variance within the observed time period can be explained. Together 83% of the variance of the analysed first differences of our VDAX sub-indices can be explained with the help of the first and second principal components. Obviously the explaining power of the principal components significantly declines from the third principal component onwards.

By displaying the eigenvalues in a graph, a form with a strong curvature at the second principal component is shown. In accordance with the well known "elbow" criterion, using the first two principal components with an explanation power of over 80% of the total variance is considered to be sufficient in describing the data set. The remaining variance can be interpreted for analytical purposes as the effect of an unsystematic error term. Figure

20.3 contains the factor loading of the first two principal components. Based on the orthogonality of the components the loading factors can be estimated using the least squares regression of the individual equations

$$x_{jt}^{c} = \sum_{l=1}^{2} b_{jl} y_{lt} + \varepsilon_t, \tag{20.3}$$

Here $\varepsilon_t$ is an independent error term.

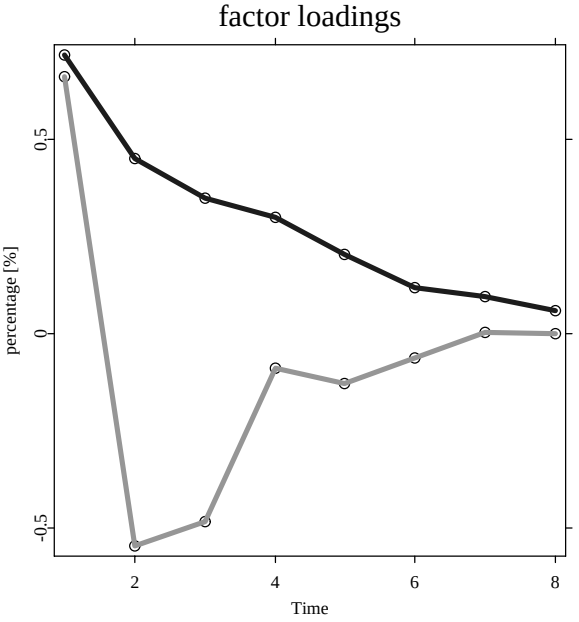


Figure 20.3: Factor loadings of the first and second principal components
SFEPCA

Based on the factor loadings it is clear that a shock to the first factor would affect the implied volatility of all times to maturity considered in a similar way, or would cause a non-parallel shift in the maturities' structure. A shock to the second principal component, on the other hand, causes a tilt of the structure curve: while at short times to maturity it causes a positive change, the longer time to maturities are influenced negatively. The absolute size of the effect of a shock decreases in both factors with the time to maturity.

| Principal Components: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| **Weekly data** | | | | | | | | |
| 18.03.96-19.12.97 | 73.85 | 91.59 | 95.09 | 97.38 | 98.80 | 99.61 | 99.88 | 100 |
| **Daily data** | | | | | | | | |
| 18.03.96-19.12.97 | 70.05 | 83.12 | 88.69 | 91.80 | 94.86 | 96.97 | 98.90 | 100 |
| **Sub-period 1** | | | | | | | | |
| 18.03.96-05.02.97 | 83.36 | 91.84 | 94.65 | 96.39 | 97.76 | 98.78 | 99.52 | 100 |
| **Sub-period 2** | | | | | | | | |
| 05.02.97-19.12.97 | 68.22 | 82.21 | 87.99 | 91.35 | 94.64 | 96.93 | 98.86 | 100 |

Table 20.3: Explained portion of the variance (in percentage) in different sub-periods

## 20.3 Stability Analysis of the VDAX's Dynamics

In order to sensibly apply the principal factors in measuring the risk of portfolios we have to study their stability over time. When the principal components and the factor loadings change significantly over time, a global analysis would not be suitable to illustrate the future variance of the implied volatility nor to judge the risks of the option portfolios with sufficient accuracy.

Our procedure considers two aspects: first whether the random portion in daily data is possibly significantly higher than in weekly data. A possible cause for this is the non-synchronous trading caused by frequent realisations of the quotes in the liquid contracts with a short time to maturity and sparsely available prices in the long running contracts. In order to distinguish the possible influences of this effect, we run our analysis analogously based on weekly data. By sufficient stability in the principal components the use of daily, and respectively, weekly data should lead to similar results.

For the second aspect we divide our data into two non-overlapping periods of equal length. Each sub-period contains $m = 220$ daily observations of the process of the differences. For each sub-period we run a principal component analysis as described above and compare the respective sizes of the eigenvalues $\hat{\lambda}_k^i, k = 1, 2$, in both sub-periods $i = 1, 2$.

As already mentioned the effect of non-synchronous trading that appears in daily data can be eliminated by using weekly data. From Table 20.3 it emerges that the explanatory power of the first principal component is slightly higher in weekly data. This is not surprising given the expected size of the error terms proportion in daily data. Overall the explanatory proportions of the variance have similar values when using weekly data. This supports the stability of the analysis method used here w.r.t. the bias due

to non-synchronous trading in daily data.

From Table 20.3 it emerges that the proportion of the variance explained by the first two principal components declines in the second sub-period. Based on this a stability test is necessary: A two sided confidence interval for the difference of the eigenvalues from both sub-periods is

$$\ln \hat{\lambda}_k^1 - 2q_\alpha \sqrt{\frac{2}{m-1}} \leq \ln \hat{\lambda}_k^2 \leq \ln \hat{\lambda}_k^1 + 2q_\alpha \sqrt{\frac{2}{m-1}}, \tag{20.4}$$

where $q_\alpha$ represents the $\alpha$ quantile of a standard normal distribution, see Härdle and Simar (2007). From this it follows that

$$| \ln \hat{\lambda}_k^1 - \ln \hat{\lambda}_k^2 | \geq 2q_\alpha \sqrt{\frac{2}{m-1}} \tag{20.5}$$

is a second test for $H_0 : \lambda_k^1 = \lambda_k^2$. Under the null hypothesis the respective eigenvalues are the same in both periods. The null hypothesis is rejected when the inequality is fulfilled for a corresponding critical value $q$. This would indicate an instability of the principal components over time.

Critical values for rejecting the null hypothesis are 0.313 (probability of error 10%), 0.373 (probability of error 5%) and 0.490 (probability of error 1%). The differences of the estimated eigenvalues are 0.667 and 1.183. Both differences are significantly larger than zero with an error probability of 1%. These results prove that the determining factors of the volatility dynamics change over time. By the determination of the risk of option portfolios it therefore appears necessary to use an adaptive method of the principal components. Here the estimation is periodically done over a moving time window and the length of the time window is adaptively set, see Cizek, Härdle and Weron (2005).

## 20.4 Measure of the Implied Volatility's Risk

The market value $P_t$ of a portfolio consisting of $w$ different options is dependent on changes of the risk free interest rate $r_t$, the prices $S_t$ of the financial underlying, the time to maturity $\tau$ and the individual implied volatilities $\sigma_I$. Changes in the portfolio value can be analytically approximated using the following Taylor approximation, where it is assumed that the options are all based on the same underlying.

$$\begin{aligned}\Delta P_t &= \sum_{u=1}^{w}\Big\{\frac{\partial V_{ut}}{\partial \sigma_I}\Delta\sigma_{I,t}+\frac{\partial V_{ut}}{\partial t}\Delta t+\frac{\partial V_{ut}}{\partial r}\Delta r_t \\ &\qquad +\frac{\partial V_{ut}}{\partial S}\Delta S_t+\frac{1}{2}\frac{\partial^2 V_{ut}}{\partial S^2}(\Delta S_t)^2\Big\} \\ \Delta P_t &= \sum_{u=1}^{w}\left\{\frac{\partial V_{ut}}{\partial \sigma_I}\Delta\sigma_{I,t}+\Theta_u\Delta t+\rho_u\Delta r_t+\Delta_u\Delta S_t+\frac{1}{2}\Gamma_u(\Delta S_t)^2\right\}\end{aligned}$$

Here $V_{ut}$ describes the price of the $u$-th option with a time to maturity $\tau_u$ at date $t$, and $\Theta_u, \rho_u, \Delta_u, \Gamma_u$ are the characteristic values described in Section 6.4 of the $u$-th option. In practice option traders often insert "Vega" positions directly. In doing so they create portfolios whose profit and loss profile can be determined by the changes in the implied volatilities of the respective options, see Taleb (1997). Portfolios of this kind are called $(\Delta, \Gamma)$ and $\Theta$ neutral. The sensitivity of the option price under consideration to the changes in the volatilities is measured by the variable $\mathcal{V}$ ("Vega" - see (6.33)).

A well known strategy in utilising the forecasted changes in the maturity structure of implied volatilities consists of buying and selling so called "Straddles" with varying maturities. A straddle is constructed by simultaneously buying ("Long Straddle") or selling ("Short Straddle") the same number of ATM Call and Put options with the same time to maturity. If a trader expects a relatively strong increase in the implied volatility in the short maturities and a relatively weaker increase in the longer maturities, then he will buy straddles with a short time to maturity and sell longer maturity straddles at a suitable ratio. The resulting option portfolio is $(\Delta, \Gamma)$ neutral and over a short time frame $\Theta$ neutral, i.e., it is insensitive with respect to losing value over time. The Taylor series given above can thus be reduced to:

$$\Delta P_t \approx \sum_{u=1}^{w}\left\{\frac{\partial V_{ut}}{\partial \sigma_I}\Delta\sigma_{I,t}^{(t)}\right\} \tag{20.6}$$

The first differences of the implied volatilities can now be given as linear combinations of the principal components. By substituting the volatility indices $\sigma_{I,t}$, which are temporally next to the actual implied volatility $\hat{\sigma}_I(\tau_u^*)$, one obtains the following representation given (20.3):

$$\Delta P_t \approx \sum_{u=1}^{w}\left\{\frac{\partial V_{ut}}{\partial \sigma_I}\left(\sum_{l=1}^{2} b_{jl}y_{lt}\right)\right\} \tag{20.7}$$

The number of principal components used in the previous expression can be reduced to the first two without any significant loss of information.

The following "Maximum Loss" (ML) concept describes the probability distribution of a short-term change in the portfolio value dependent on changes in the value of the underlying factors. The change in value of a $(\Delta, \Gamma)$ neutral option portfolio is substantially determined by the changes in the implied volatilities of the options contained in the portfolio. To determine the "Maximum Loss" it is necessary to have an adequate exact representation of the future distribution of the changes to the volatility of the options with varying time to maturity.

The "Maximum Loss" is defined as the largest possible loss of a portfolio that can occur over a specific factor space $A_t$ and over a specific holding period $\tau$. The factor space $A_t$ is determined by a closed set with $\mathrm{P}(A_t) = \alpha$. Here $\alpha$ is set to 99% or 99.9%. The ML definition resembles at first sight the "Value-at-Risk" Definition (see Chapter 16). There is, however, an important difference between the two concepts: In calculating the "Value-at-Risk" the distribution of the returns of the given portfolio must be known, whereas the ML is defined directly over the factor space and thus has an additional degree of freedom, see Studer (1995).

In our analysis we have divided the maturity structure of the implied volatilities into two principal components, which explain a considerable portion of the variability of the structure curve. Thus the first two principal components represent the risk factors used in the ML model. The profit and loss profile of each portfolio held is determined by the corresponding changes in the risk factors using a suitable valuation model. In order to obtain this, a valuation of the underlying portfolios must theoretically occur for every point in the factor space. In the practical application the factor space is probed over a sufficiently small grid of discrete data points $y_1^z (z = 1, ..., N_1)$, during which the other risk factor is held constant in each case. Due to the orthogonality properties of the principal components, the profit and loss function $PL()$ is additive with $PL(y_1^{z_1}, y_2^{z_2}) = PL(y_1^{z_1}) + PL(y_2^{z_2})$.

Under the assumption of multivariate, normally distributed principal components confidence intervals can be constructed for the "Maximum Loss" over the total density

$$\varphi_2(y) = \frac{1}{(2\pi)\sqrt{\det \Lambda_2}} \exp(-\frac{1}{2} y^\top \Lambda_2^{-1} y), \tag{20.8}$$

with $y = (y_1, y_2)^\top$. Here the matrix $\Lambda_2$ represents the $2 \times 2$ diagonal matrix

of the eigenvalues $\lambda_k$, $k = 1{,}2$. The random variable $y^\top \Lambda_2^{-1} y = X_1^2 + X_2^2$ has a Chi-square distribution. The confidence interval for an existing portfolio is then $A_t = \{y; y^\top \Lambda_2^{-1} y \leq c_\alpha\}$, $c_\alpha$, where $c_\alpha$ is the $\alpha$ quantile of a random variable with a Chi-square distribution and 2 degrees of freedom.

## 20.5 Recommended Literature

The presentation of this chapter closely follows the work of Fengler, Härdle and Schmidt (2002). The principal components analysis is applied to the changes of implied volatilities for fixed ranges of days to maturity by Skiadopoulos, Hodges and Clewlow (1998) who find two principal components can already sufficiently explain the dynamics of smiles. The conditions to ensure the absence of arbitrage in the volatility models are derived by Schönbucher (1999). Furthermore, Härdle and Hafner (2000) show that the prices of out-of-the-money options strongly depend on volatility features such as asymmetry.

Cizek et al. (2005) develop an adaptive method of estimation which does not use any information about the time homogeneity of the observed process. It can be used to estimate the principal components. For the effect of the implied volatilities changes on the dynamic hedging of exotic and complex options we refer to Taleb (1997).

# Part III

# Selected Financial Applications

# 21 Nonparametric Estimators for the Probability of Default

The estimation of the probability of default based on information on the individual customer or the company is an important part of credit screening, i.e., judging the credit standing. It is essential for the establishment of a rating or for measuring credit risk to estimate the probability that a company will end in financial difficulties within a given period, for example, one year. Also, here nonparametric applications prove to be flexible tools in estimating the desired default probability without arbitrary assumptions. In this chapter we will give a brief overview of the various approaches for non- and semiparametric estimates of conditional probabilities.

## 21.1 Logistic Regression

In order to judge the credit standing of a customer a series of data are in general available. For consumer credit there are, for example, in Müller (2000): *level of credit, age of the customer, duration of credit* as well as information on whether the customer is *unemployed* or not and whether there were problems in the past with *repaying loans*. For the insolvency prognoses for a small company relevant information would, for example, be in Anders (1997): *age of the business, sales development* from the recent past, *educational degree of the entrepreneur, type of business* and information on *liability*.

Some influential values are quantitative, such as credit volume and sales development. Others are qualitative in nature and must be transformed into numbers for estimating the default probability. For dichotomic characteristics (unemployed, employed, limited liability, unlimited liability) indicator variables are set with values of 0 and 1. For characteristics with $d > 2$ possibilities and for categorical values $d - 1$ dummy variables are introduced, which also take on the value of 0 or 1. Coding the characteristics numerically the *type of business* and three clarifying variables *trade, processed business, other* are considered for which two Dummy variables, $Z_1, Z_2$, are used where $Z_1 = 1$ ($Z_2 = 1$) if and only if the type of business is *trade (processed busi-*

*ness).* When $Z_1 = Z_2 = 0$, the firm considered belongs to one of the *other* types of business, for example, *services.* The case $Z_1 = Z_2 = 1$ cannot occur.

If the values of the qualitative characteristics are hierarchically ordered, then it is possible to represent them with an integer valued random variable. The personal impression of the processor in the bank of the economic situation of a company: *very good, good, satisfactory, poor, very poor* can, for example, be transformed into a number scale: *1, 2, 3, 4, 5.* Here one must be certain that every monotone transformation, i.e., where the order remains consistent, produces a different numerical code that can be used with the same justification. Instead of *1, 2, 3, 4, 5* one could also use *0, 1, 3, 6, 10* for instance. Using parametric applications such as the logistic regression one should specify the arbitrary setting of a numerical scale for the hierarchical characteristics. Through a monotone transformation of the scale better estimates can eventually be obtained for the default probabilities. Adequately flexible nonparametric and semi-parametric applications, in contrast, automatically choose a suitable scale.

In order to estimate the default probability of a credit, given the information available at the time the decision is made, we assume a random sample $(X_1, Y_1), \ldots, (X_n, Y_n)$ is independent, identically distributed. $X_j \in \mathbb{R}^d$ stands for the information available at the time the credit is issued to the $j$-th customer, where qualitative characteristics are already transformed into numerical values as described above. $Y_j \in \{0, 1\}$ is the indicator variable of the credit: it has a value of 0 when the loan can be paid back without any problems and 1 when the credit partially or completely defaulted. The default probability that is to be estimated is the conditional probability that $Y_j = 1$, given $X_j = x$ :

$$\pi(x) = \mathrm{P}(Y_j = 1 | X_j = x),\ x \in \mathcal{X},$$

where $\mathcal{X} \subset \mathbb{R}^d$ represents the value space of $X_j$.

Since $\pi(x)$ only takes on the values between 0 and 1 given that it is a probability, linear regression models cannot be used for the function estimator. The class of generalised linear models (GLM) can, however, be used to estimate the probabilities. Here it is assumed that

$$\pi(x) = G(\beta_0 + \sum_{i=1}^{d} x_i \beta_i) = G(\beta_0 + \beta^\top x).$$

$G : \mathbb{R} \to [0, 1]$ is a known function that only takes on a value between 0 and 1, the real valued parameters $\beta_0, \ldots, \beta_d$ are unknown and need to be estimated.

For the special case that $G$ is chosen to be a logistic function $\psi$:

$$G(t) = \psi(t) = \frac{1}{1 + e^{-t}},$$

we obtain the model of the *logistic regression*: Given $X_1, \ldots, X_n$, the credit indicators $Y_1, \ldots, Y_n$ are independent Bernoulli random variables with parameters $\psi(\beta_0 + \beta^\top X_1), \ldots, \psi(\beta_0 + \beta^\top X_n)$. The conditional likelihood function is thus

$$L(\beta_0, \ldots, \beta_d) = \Pi_{j=1}^n [Y_j \psi(\beta_0 + \beta^\top X_j) + (1 - Y_j)\{1 - \psi(\beta_0 + \beta^\top X_j)\}].$$

Since $Y_j$ only takes on a value between 0 and 1, the corresponding conditional log-likelihood function is

$$\log L(\beta_0, \ldots, \beta_d) = \sum_{j=1}^{n} [Y_j \log \psi(\beta_0 + \beta^\top X_j) + (1 - Y_j) \log\{1 - \psi(\beta_0 + \beta^\top X_j)\}].$$

Through maximising $L$ or $\log L$ one obtains the maximum likelihood estimator $\hat{\beta}_0, \ldots, \hat{\beta}_d$ of $\beta_0, \ldots, \beta_d$ and thus the maximum likelihood estimator for the default probability in the logistic regression model:

$$\hat{\pi}(x) = \psi(\hat{\beta}_0 + \hat{\beta}^\top x).$$

## 21.2 Semi-parametric Model for Credit Rating

The logistic regression model for the estimate of the conditional probability suffers under the same restrictions as the linear regression model when estimating the general functions. In order to avoid the dependence on the special parametric form of the model and to gain more flexibility in the function estimation it is recommended that $\pi(x)$ is estimated nonparametrically, for example, with the LP-method given in (14.4) and (14.7). In doing this, however, it is not guaranteed that the function estimator will lie between 0 and 1. In order to enforce this possible, as was carried out in the previous section, we transform the value space of the estimated function to the interval [0,1] using a given function $G$:

$$\pi(x) = G(m(x))$$

where $m(x)$ is an arbitrary real valued function that can be estimated nonparametrically. For the estimate of the default probabilities the local smoothing methods are less suitable for two reasons. First of all $x$ is often high dimensional in the application, for example, after adding the necessary dummy

variables in the example considered by Müller and Rönz (2000) it has a dimension of 61, that even by larger local neighbourhoods of $x$ of the random sample, over which the estimation occurs, there are either too few observations or too large to produce a reliable estimate of $m(x)$. This problem can be solved by restricting ourselves to additive models

$$\pi(x) = G(\sum_{i=1}^{d} m_i(x_i)),$$

where $m_1(u), \ldots, m_d(u)$ are arbitrary functions of the one-dimensional variable $u$. It is however more critical that many of the coordinates of $x$ take on a value of 0 or 1 in the credit rating, since they represent, from the very beginning, dichotomic characteristics or have been added as dummy variables for the unordered qualitative characteristics. Local smoothing functions would be suitable based on their underlying philosophy, but mainly for estimating functions with continuous arguments.

A combination of nonparametric and parametric applications offers the possibility of using the flexibility of the nonparametric method by credit rating, Müller and Rönz (2000). In doing so the influential variables are not combined in a random vector $X_j$, but are separated into two random vectors $X_j \in \mathbb{R}^p, Z_j \in \mathbb{R}^q$. The coordinates of $Z_j$ represent several chosen exclusive quantitative characteristics and eventual hierarchical qualitative characteristics with sufficiently accurate subdivided value spaces. All remaining characteristics, especially the dichotomic and the dummy variables of unordered qualitative characteristics, are combined in $X_j$. In order to estimate the default probability we consider a generalised partial linear model (GPLM = generalised partial linear model) :

$$\mathrm{P}(Y_j = 1 | X_j = x, Z_j = z) = \pi(x, z) = G(\beta^\top x + m(z)).$$

$G$ is again a known function with values between 0 and 1, for example, the logistic function $\psi$. $\beta_1, \ldots, \beta_p$ are unknown parameters, $m$ is an arbitrary, unknown function that can contain an additive constant and thus can make an additional parameter $\beta_0$ superfluous. In an extensive case study Müller (2000) has shown that the additional flexibility from the nonparametric part $m(z)$ of the model results in a better estimate of the default probability than a pure parametric logistic regression.

There are various algorithms for estimating $\beta$ and $m(z)$, for example the profile likelihood method from Severini and Wong (1992) and Severini and Staniswallis (1994) or the back-fitting method from Hastie and Tibshirani (1990). Essentially they use the fact that for the known function $m(z)$ of

the parameter vector $\beta$ can be estimated through maximisation of the log-likelihood function analog to the logistic regression

$$\log L(\beta) = \sum_{j=1}^{n} [Y_j \log G(\beta^\top X_j + m(Z_j)) + (1 - Y_j) \log\{1 - G(\beta^\top X_j + m(Z_j))\}]$$

and for known $\beta$ the function $m(z)$ can be estimated with local smoothing analog to the LP-Method (14.4), (14.7). Both of these optimisation problems are combined in an iterative numerical algorithm.

**Example 21.1**
*As an example we consider the rating of consumer credit already referred to above that Müller (2000) had carried out with a GPLM method. The data represent a part of the extensive random sample, which is described in detail by Fahrmeir and Tutz (1994). We use a total of* $n = 564$ *observations, in which 24.3% of the cases have a problem with repaying the credit (*$Y_j = 1$*). From the 5 influential variables considered, two are dichotomic; they indicate whether the customer is unemployed or not (*$X_{j1}$*) and whether the customer has had credit problems in the past or not (*$X_{j2}$*). The remaining three variables are quantitative: the duration of the credit (*$X_{j3}$ *with values between 4 and 72 months), the level of the credit (between DM 338 and DM 15653) and the age of the customer (between 19 and 75 years). We will take the logarithm of the last two variables and transform them linearly so that they take on a value in the interval* $[0, 1]$*. The data points, as can be seen in Figure 21.1, are dispersed comparatively homogenous over a part of the plane, which makes the local smoothing easier. These transformed variables are called* $Z_{j1}$ *and* $Z_{j2}$*. We fit a GPLM*

$$\begin{aligned} \mathrm{P}(Y_j = 1 \quad | \quad & X_{j1} = x_1, X_{j2} = x_2, X_{j3} = x_3, Z_{j1} = z_1, Z_{j2} = z_2) \\ & = \psi(\sum_{k=1}^{3} \beta_k x_k + m(z_1, z_2)) \end{aligned}$$

*to the data and obtain the estimates (the corresponding standard deviation is given in parentheses)*

$$\beta_1 = 0.965\,(0.249),\ \beta_2 = 0.746\,(0.237),\ \beta_3 = -0.0498\,(0.0115).$$

*The probability of default on the credit increases when the customer is unemployed or if the customer has had repayment problems in the past; this however decreases with the duration of the credit. The dependence on the*

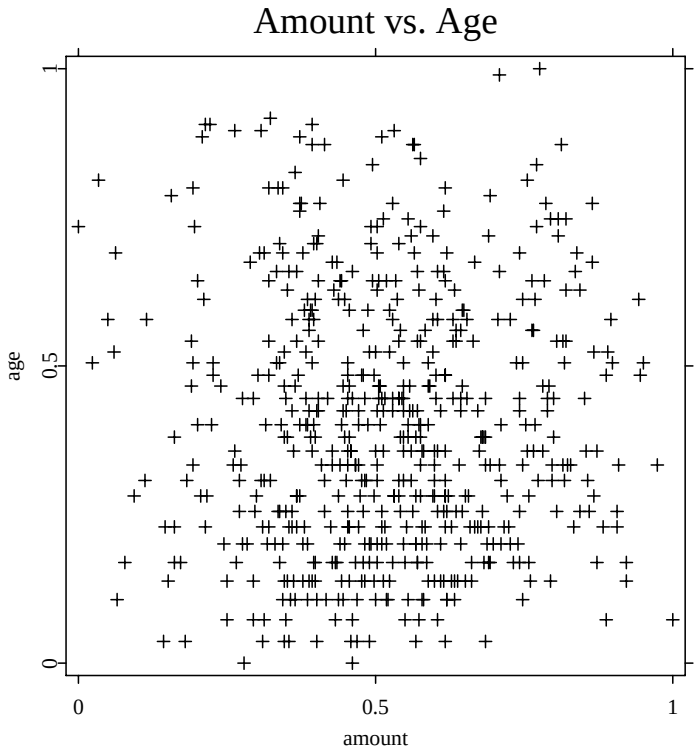


Figure 21.1: The scatter plot of the transferred variables: level of credit and age of the customers. SFEgplm

*transformed credit levels and ages are nonparametrically estimated. From Figure 21.2 it is obvious that the estimated function $\hat{m}(z_1, z_2)$ is clearly nonlinear with a maximum by the average value of the credit level and age. The decrease in the probability of default by high levels of credit can be explained by the fact that the random sample contains only those credits that have actually been given and that the processor was essentially reluctant to grant high levels of credit when the customer appeared to be unreliable. This effect, which is caused by credit ratings from the past, occurs on a regular basis in credit assessment, even if a systematic, model based method is not used, which excludes the credit screening of extreme risks from the very beginning and thus mean that these ratings no longer appear in the data. This must therefore be considered when interpreting and applying a model.*

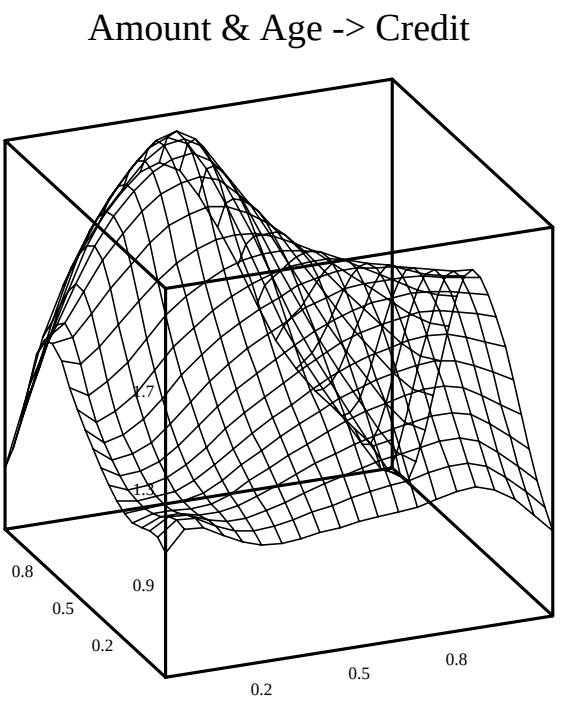


Figure 21.2: The estimated function with respect to level of credit and age of the customers. SFEgplm

## 21.3 Credit Ratings with Neural Networks

As with nonparametric fitting of financial time series models to data, the neural network also provides an alternative to local smoothing, as with the LP method, in estimating default probabilities. The logistic regression function $\pi(x) = \psi(\beta_0 + \beta^\top x)$ is nothing more than a function defined by a neural network with only one neuron in a hidden layer, when the logistic function $\psi$ is chosen as a transfer function. Through the combination of several neurons in one or more hidden layers default probabilities can be estimated, as with the nonparametric regression analysis, with flexibility. In order to obtain estimates between 0 and 1, it is necessary to represent the function $\nu_H(x;\delta)$ given by (19.1), for example, with a function $G$ over the interval [0,1]. We restrict ourselves to one hidden layer with $H$ neurons and choose $G = \psi$, so

that the default probability given by the neuron network has the form

$$\pi_H(x;\vartheta) = \psi(v_0 + \sum_{h=1}^{H} v_h\, \psi(w_{oh} + \sum_{j=1}^{d} w_{ih}x_i)),$$

where $\vartheta$ once again represents the parameter vector built from $v_h$ and $w_{ih}$, $0 \leq i \leq d$, $0 \leq h \leq H$. To estimate the network weights from the data we will not use the least squares method, which makes sense for the regression model with normally distributed residuals, but instead we will maximise the log-likelihood function

$$\log L(\vartheta) = \sum_{j=1}^{n} [Y_j \log \pi_H(X_j;\vartheta) + (1 - Y_j) \log\{1 - \pi_H(X_j;\vartheta)\}]$$

following the procedure used in the logistic regression. By substituting $\hat{\vartheta}_n$ in the estimator we obtain an estimator for the default probability

$$\hat{\pi}(x) = \pi_H(x;\hat{\vartheta}_n).$$

In order to obtain an especially simple model with fewer parameters, Anders (1997) trivially modified the method for the insolvency prognoses for small and middle sized firms and assumed a default probability of the form

$$\pi_H^l(x;\vartheta) = \psi(\beta^\top x + v_0 + \sum_{h=1}^{H} v_h \psi(w_{oh} + \sum_{i=1}^{d} w_{ih}x_i)),$$

which has obvious similarities to the general partial linear model, besides the fact that here a part of or all of the influential variables, i.e., the coordinates of $x$, can appear in linear as well as in nonparametric portions. The linear term $\beta^\top x$ can be interpreted as the value of an additional neuron whose transfer function is not the logistic function $\psi(t)$, but the identity $f(t) \stackrel{\text{def}}{=} t$. Estimating the network from the application of a model selection technique, utilised to find the insolvency probability, is surprisingly easy. In addition to a linear term it contains only one single neuron ($H = 1$). From the 6 input variables only 4 contribute to the linear part (age of the business, sales development, indicator for limited liability, dummy variable for processed business); this means the other two coefficients $\beta_i$ are 0, and only 3 (Dummy variables for processed business and for trade, indicator variable for educational degree of entrepreneur) contribute to the sigmoid part, meaning that the corresponding weights $w_{i1}$ are 0. With this simple model using a validation data set, which is not used to estimate the parameters, a ratio of the correct identifications of 83.3% was obtained for the insolvencies and of 63.3 % for the solvent companies.

# 22 Credit Risk Management

Credit risk management is an important issue in banking. In this chapter we give an overview of the models for calculating the default risk exposure of a credit portfolio. The primary goal of these models is to help credit analysts define whether the loan should be issued, which risk premia is appropriate, and how much capital should be directed to the loss reserve account. We follow closely the presentation of Bluhm, Overbeck and Wagner (2002).

## 22.1 Basic Concepts

**Definition 22.1 (Loss function)**
*The loss fraction in case of default is called loss given default (LGD). The exposure at default in a considered time period is abbreviated to EAD. The loss of an obligor is thus defined by the following loss function:*

$$\tilde{L} = EAD \times LGD \times L$$

*with $L = \mathbf{1}(D)$. Here $D$ stands for the default event of an obligor in a given time period, (e.g. one year). $P(D)$ is the probability of the event $D$.*

**Definition 22.2 (Expected Loss)**
*The expected loss (EL) is defined as:*

$$EL = \mathsf{E}(\tilde{L}) = EAD \times LGD \times P(D).$$

For the sake of simplicity we assume here (and thereafter) that EAD and LGD are deterministic, therefore implying their independence from the default event. However violating these assumptions leads to a more specific and realistic model.

To calculate $EL$ we need to find default probabilities which could be inferred either from credit ratings or from market prices of defaultable bonds or credit derivatives. According to the first approach we use data on default frequencies for different rating classes to perform a mapping from the ratings'

space into the default probabilities' space. For further details please refer to Bluhm et al. (2002). A good description of the market data approach can be found in Li (1998).

The expected loss $EL$ defines the necessary loss reserve that a bank must hold as insurance against a default. In addition to the expected loss the bank should have a cushion to cover unexpected losses.

**Definition 22.3 (Unexpected Loss)**
*The unexpected loss (UL) is defined as:*

$$UL = \sqrt{Var(\tilde{L})} = \sqrt{Var(EAD \times LGD \times L)}$$

*with* $Var(L) = P(D)(1 - P(D))$.

So far we have considered the loss estimates for a single obligor. Now assume we have a credit portfolio consisting of $m$ loans.

**Definition 22.4 (Portfolio Loss)**
*The expected portfolio loss is defined by the following random variable:*

$$\tilde{L}_{PF} = \sum_{i=1}^{m} \tilde{L}_i = \sum_{i=1}^{m} EAD_i \times LGD_i \times L_i$$

*with* $L_i = \mathbf{1}(D_i)$.

Analogously to the single obligor case we can calculate $EL_{PF}$ and $UL_{PF}$:

$$EL_{PF} = \sum_{i=1}^{m} EL_i = \sum_{i=1}^{m} EAD_i \times LGD_i \times \mathrm{P}(D_i)$$

$$UL_{PF} = \sqrt{\sum_{i,j=1}^{m} EAD_i \times EAD_j \times LGD_i \times LGD_j \times \mathrm{Cov}(L_i, L_j)}.$$

It is possible to rewrite the covariance term as following: $\mathrm{Cov}(L_i, L_j) = \sqrt{\mathrm{Var}(L_i) \times \mathrm{Var}(L_j) \times \rho_{ij}}$. We now obviously face the problem of the unknown default correlations $\rho_{ij}$. One could assume that loss variables are uncorrelated but this severely contradicts our empirical observations; defaults are likely to happen jointly so that the correlation between obligors becomes the main driver of credit risk and the key issue in credit modelling. Additionally we will introduce the models which incorporate the statistical techniques for calibrating default correlations.

## 22.2 The Bernoulli Model

In the preceding section we implicitly introduced the Bernoulli loss variable defined as $L_i \sim B(1; p_i)$, with $L_i$ being the default variable of obligor $i$, i.e. loss is generated with probability $p_i$ and not generated with probability $(1 - p_i)$. The fundamental idea in the modelling of joint defaults is the randomisation of the involved default probabilities. While in our previous analysis we considered extracted market data or ratings default probabilities, now we assume that the loss probabilities are random variables that follow some distribution $F$ within $[0,1]^m$: $P = (P_1, \ldots, P_m) \sim F$.

We assume that Bernoulli loss variables $L_1, \ldots, L_m$ are independent conditional on a realisation $p = (p_1, \ldots, p_m)$ of vector $P$. The joint distribution of the loss function is then:

$$\mathrm{P}(L_1 = l_1, ..., L_m = l_m) =$$

$$\int_{[0,1]^m} \prod_{i=1}^{m} p_i^{l_i} (1 - p_i)^{1-l_i} \, dF(p_1, ..., p_m), \tag{22.1}$$

where $l_i \in \{0, 1\}$. The first and second moments of the single losses $L_i$ are:

$$\mathsf{E}(L_i) = \mathsf{E}(P_i), \quad \mathrm{Var}(L_i) = \mathsf{E}(P_i)\{1 - \mathsf{E}(P_i)\}$$

The covariance of single losses is given by:

$$\mathrm{Cov}(L_i, L_j) = \mathsf{E}(L_i, L_j) - \mathsf{E}(L_i)\,\mathsf{E}(L_j) = \mathrm{Cov}(P_i, P_j) \tag{22.2}$$

The correlation for two counterparties' default is:

$$\mathrm{Corr}(L_i, L_j) = \frac{\mathrm{Cov}(P_i, P_j)}{\sqrt{\mathsf{E}(P_i)\,\{1 - \mathsf{E}(P_i)\}}\sqrt{\mathsf{E}(P_j)\,\{1 - \mathsf{E}[P_j]\}}}. \tag{22.3}$$

Thus we have succeeded in expressing the unknown default correlations in terms of covariances of the $F$ distribution. Later in this chapter (p. 457) we will illustrate how to obtain an appropriate specification for the distribution of default probabilities and consequently solve the default correlations.

A major simplification is possible if one assumes an equal default probability $P_i$ for all obligors. It is suitable for the uniform portfolios with loans of comparable size and with similiar risk characteristics. In this case (22.1) simplifies to

$$\mathrm{P}(L_1 = l_1, ..., L_m = l_m) = \int_0^1 p^k (1 - p)^{m-k} \, dF(p) \tag{22.4}$$

where $k = \sum_{i=1}^{m} l_i$ is the number of defaults in the credit portfolio. Note that $EL$ equals:

$$\overline{p} = \int_0^1 p \, dF(p) \tag{22.5}$$

Therefore the default correlation between two different counterparties equals:

$$\rho_{ij} = \mathrm{Corr}(L_i, L_j) =$$

$$\frac{\mathrm{P}(L_i = 1, L_j = 1) - \overline{p}^2}{\overline{p}(1-\overline{p})} = \frac{\int_0^1 p^2 dF(p) - \overline{p}^2}{\overline{p}(1-\overline{p})}. \tag{22.6}$$

Formula (22.2) shows that the higher volatility of $P$ corresponds to the higher default correlation. Since the numerator of (22.2) equals $\mathrm{Var}(P) \geq 0$ the default correlation in the Bernoulli model is always positive and cannot mimic negative default correlation.

## 22.3 The Poisson Model

Another common approach to joint default modelling is the assumption of the Poisson-distributed loss variable $L_i$ with intensity $\Lambda_i$. This means that $L_i \sim Pois(\lambda_i)$, $p_i = \mathrm{P}(L_i \geq 1)$, $L_i \in \{0, 1, 2, \ldots\}$ modelling the fact that multiple defaults of one obligor $i$ may occur. Analogously to the Bernoulli mixture model we not only assume the loss variable vector $L$ but also the intensity vector $\Lambda = (\Lambda_1, \ldots, \Lambda_m)$ to be random: $\Lambda \sim F$ within $[0, \infty)^m$. Also assume that $L_1, \ldots L_m$ (conditional on a realisation of $\Lambda$) are independent. The joint distribution of $L_i$ is given:

$$\mathrm{P}(L_i = l_i, ..., L_i = l_i)$$

$$= \int_{[0,\infty)^m} e^{-(\lambda_1 + ... + \lambda_m)} \prod_{i=1}^{m} \frac{\lambda_i^{l_i}}{l_i!} \, dF(\lambda_1, ..., \lambda_m), \tag{22.7}$$

Similar to the Bernoulli case, we have for $i = 1, ..., m$ :

$$\mathsf{E}(L_i) = \mathsf{E}(\Lambda_i)$$

$$\mathrm{Var}(L_i) = \mathrm{Var}\left\{\mathsf{E}(L_i|\Lambda)\right\} + \mathsf{E}\left\{\mathrm{Var}(L_i|\Lambda)\right\} = \mathrm{Var}(\Lambda_i) + \mathsf{E}(\Lambda_i). \tag{22.8}$$

The correlation is then given:

$$\mathrm{Corr}(L_i, L_j) = \frac{\mathrm{Cov}(\Lambda_i, \Lambda_j)}{\sqrt{\mathrm{Var}(\Lambda_i) + \mathsf{E}[\Lambda_i]}\sqrt{\mathrm{Var}(\Lambda_j) + \mathsf{E}[\Lambda_j]}}. \tag{22.9}$$

Like in the Bernoulli Model we can express the default correlation through the covariances of the intensity vector distribution $F$. For the uniform portfolios we could assume a single distribution for all obligors. The analogue of (22.2) is then:

$$\mathrm{Corr}(L_i, L_j) = \frac{\mathrm{Var}(\Lambda)}{\mathrm{Var}(\Lambda) + \mathsf{E}(\Lambda)}. \tag{22.10}$$

This formula is especially intuitive if we look at it from a dispersion point of view. The dispersion of a distribution is its variance to mean ratio. The dispersion of a Poisson distribution is equal to 1. Using dispersion, we get the following formula:

$$\mathrm{Corr}(L_i, L_j) = \frac{D[\Lambda]}{D[\Lambda] + 1}. \tag{22.11}$$

We therefore conclude: an increase in dispersion will increase the mixture effect, which strengthens the dependence between obligor's defaults.

**Bernoulli vs. Poisson**
Comparing Bernoulli with Poisson distribution of the default risk, we see that a higher default correlation in Bernoulli distribution than in Poisson distribution always exists. In other words even in cases where the mean of Bernoulli matches the Poisson distribution, the Poisson variance will always exceed the variance of Bernoulli. The higher default correlations result in fatter tails of the corresponding loss distributions.

## 22.4 The Industrial Models

**CreditMetrics™ and KMV Models**
Two well-known factor models applied to the major financial institutions are $CreditMetrics^{TM}$ and KMV. Both models belong to the Bernoulli class and imply only two possible outcomes — default or survive. Default of an obligor $i$ occurs if the value of the obligor's assets $A_T^{(i)}$ in a valuation horizon $T$ falls below a threshold value $C_i$, often interpreted as the value of the obligor's liabilities.

$$L_i = \mathbf{1}\{A_T^{(i)} < C_i\} \sim B\{1; P(A_T^{(i)} < C_i)\} \tag{22.12}$$

Thus $A_T$ can be regarded as a latent variable, which drives the default event implicitly replacing the notion of default correlation for the asset correlation. How is the correlation matrix of the latent variables defined? The answer lies in the basic assumption of both models, according to which the asset value dynamics relate to the changes in some common factors reflecting economic issues. Therefore asset correlations between obligors are induced exclusively by the correlation between the respective composite factors denoted by $Y_i$. In the typical model parametrization the latent variables are presented in the form of standardised asset log-returns:

$$r_i = \frac{\tilde{r}_i - \mathsf{E}(\tilde{r}_i)}{\sqrt{\mathrm{Var}(\tilde{r}_i)}} \qquad \text{with} \qquad \tilde{r}_i = \log(A_T^{(i)}/A_0^{(i)}).$$

Suppose that the standardised log return of the asset value can be written as:

$$r_i = R_i Y_i + \varepsilon_i. \tag{22.13}$$

Here $Y_i$ represents a weighted sum of many industry and country indices (composite factor). From the simple regression analysis we conclude that $R_i^2$ defines how much the volatility of $r_i$ can be explained by the volatility of $Y_i$ and therefore it stands for the systematic risk of the obligor $i$. Respectively $\varepsilon_i$ is the firm-specific effect.

The core assumption of $CreditMetrics^{TM}$ and KMV models is the multivariate normal (Gaussian) distribution of the latent variables $r_i$:

$$\begin{aligned} r_i &\sim N(0,1) \\ Y_i &\sim N(0,1) \\ \varepsilon_i &\sim N(0, 1-R_i^2) \end{aligned}$$

In this case we can rewrite (22.12) as:

$$L_i = \mathbf{1}\{r_i < c_i\} \tag{22.14}$$

where $c_i$ is the threshold corresponding to $C_i$ after replacing $A_T$ for the standardised log returns $r_i$. Using (22.13) we can rewrite the threshold condition $r_i < c_i$ as $\varepsilon_i < c_i - R_i Y_i$. Because $r_i \sim N(0,1)$, from $p_i = \mathrm{P}(r_i < c_i)$ we obtain

$$c_i = \Phi^{-1}(p_i).$$

After standardising of $\varepsilon_i$ the threshold condition changes to:

$$\frac{\varepsilon_i}{\sqrt{1-R_i^2}} < \frac{\Phi^{-1}(p_i) - R_i Y_i}{\sqrt{1-R_i^2}}. \tag{22.15}$$

On the right hand side of (22.15) $Y_i$ is the only stochastic element. We therefore obtain (conditional on $Y_i = y$)

$$p_i(y) = \Phi\{\frac{\Phi^{-1}(p_i) - R_i y}{\sqrt{1-R_i^2}}\}. \tag{22.16}$$

Transforming this into the Bernoulli mixture setting yields

$$P(L_1 = l_1, \ldots, L_m = l_m)$$

$$= \int_{[0,1]^m} \prod_{i=1}^{m} q_i^{l_i} (1-q_i)^{1-l_i} dF(q_1, ..., q_m).$$

Now we are able to specify the probability distribution function:

$$F(q_1, ..., q_m) = N_m(\mu, \Gamma)$$

where $\mu = (p_1^{-1}(q_1), \ldots, p_m^{-1}(q_m))^\top$ and $\Gamma$ is the asset correlation matrix of the log returns $\tilde{r}_i$.

The described modelling framework belongs to the KMV model. Though being based on the same assumptions, $CreditMetrics^{TM}$ differs from the KMV mainly in two issues: it uses equity instead of asset value process and it incorporates a slightly different approach to defining composite factors. For further information on the model please refer to $CreditMetrics^{TM}$ Technical Document.

**CreditRisk$^+$ Model**
In contrast to the KMV and $CreditMetrics^{TM}$ the $CreditRisk^+$ represents the class of Poisson mixture models. Also instead of common factors it introduces the notion of sectors and is therefore sometimes called "sector model". A sector plays a similiar role in the model as compared to the factor but unlike the factor which has a certain economic interpretation, the sector may reflect any issues that have a systematic effect on the obligor's economic performance. Each sector is assumed to be Poisson distributed random variable with gamma distributed intensity $\Lambda^{(s)}$, where the variables $\Lambda^{(1)}, \ldots, \Lambda^{(m_S)}$ are assumed to be independent. For the reader's convenience the gamma distribution pdf is stated here:

$$\{\beta^\alpha \Gamma(\alpha)\}^{-1} x^{\alpha-1} \exp(-x/\beta).$$

The mean and variance of a gamma distributed random variable $\Lambda$ are:

$$\begin{aligned} \mathsf{E}\,\Lambda &= \alpha\beta \\ \mathrm{Var}(\Lambda) &= \alpha\beta^2. \end{aligned} \tag{22.17}$$

Consider a credit portfolio of $m$ loans of $m$ different obligors. In $CreditRisk^+$ a weight $\omega_{is}$ of default intensity of obligor $i$ to the systematic default risk of sector $s$ is given, with $\sum_{s=1}^{S} \omega_{is} = 1$. The basic idea is that the risk of the sector $s \in \{1, \ldots, m_S\}$ is driven by two factors. The first driver is the mean default intensity:

$$\lambda_{(s)} = \mathsf{E}(\Lambda^{(s)}) = \alpha_s \beta_s. \tag{22.18}$$

The second driver is the default intensity's volatility:

$$\sigma^2_{(s)} = \mathrm{Var}(\Lambda^{(s)}) = \alpha_s \beta_s^2. \tag{22.19}$$

The default risk of obligor $i$ is modelled as well by a mixed Poisson variable $L$ with the default intensity $\Lambda_i$ with mean value $\mathsf{E}(\Lambda_i) = \lambda_i$. From the probability theory we know that for Poisson distributed variables with small intensity $\lambda_i$ the following expression is true:

$$p_i = \mathrm{P}(L_i \geq 1) = 1 - e^{-\lambda_i} \approx \lambda_i. \tag{22.20}$$

Thus we can calibrate the default intensity from the obligor's one-year default probability. The sector parametrization of $\Lambda_i$ is given:

$$\Lambda_i = \sum_{s=1}^{S} \omega_{is} \lambda_i \frac{\Lambda^{(s)}}{\lambda_{(s)}} \tag{22.21}$$

This means that obligors admit a common source of systematic default risk (i.e. they are correlated), if and only if there is at least one sector with a positive weight with respect to the considered obligors. The formula (22.21) is consistent with the assumption that $\lambda_i$ equals the expected default intensity of obligor $i$.

According to the formula (22.20), the conditional default intensity of obligor $i$ that arises from the realisations $\theta_1, ..., \theta_{m_S}$ of the sector default intensities $\Lambda^{(1)}, ..., \Lambda^{(m_S)}$ generates a conditional one-year default probability:

$$\begin{aligned} p_i(\theta_1, ..., \theta_{m_S}) &= \mathrm{P}(L_i \geq 1 | \Lambda^{(1)} = \theta_1, ..., \Lambda^{m_S} = \theta_{m_S}) \\ &= 1 - \exp(-\lambda_i \sum_{s=1}^{m_S} w_{is} \theta_s / \lambda_{(s)}) \end{aligned} \tag{22.22}$$

To make the idea behind this model more pellucid, we can relate the notion of sector to the notion of factor and the obligor's default intensity $\Lambda_i$ — to the default probability $p_i$ in the respective factor models. The latter expression could be related then to the formula (22.16) in the Bernoulli framework of the KMV model. As in the previous section, we will proceed now to the problem of the portfolio loss distribution specification. Assuming we know the distribution of defaults in every single sector we can obtain the portfolio's default distribution as the convolution of the sector distributions due to the independence of the sector variables $\Lambda^{(1)}, \ldots, \Lambda^{(m_S)}$. For details on how to find the sectors' default distributions please refer to Bluhm et al. (2002) or directly to $CreditRisk^+$ Technical Document.

**Other Models**

Among other "best-practice" industrial models is the CreditPortfolioView (CPV) generated from a macroeconomic approach. Before moving on we should explain the notion of migration matrix used in the model. Migration matrix contains probabilities for the rated bonds of being downgraded, upgraded or retaining the same grade in a set time period. For every rating class the number of transitions and respectively the probabilities of these transitions will sum up to the total number of possible ratings within a certain rating system.

CPV is a rating-based portfolio model that incorporates the dependence of default and migration probability on business cycles. The model uses Monte-Carlo simulation to generate migration matrices and thus generates macro-scenarios. The losses arising in every possible economic environment are then tabulated and, based on these losses the portfolio loss distribution is constructed. A significant advantage of this model is in its ability to allow for the fat tails in loss distribution, which can be observed emirically in the periods of deep depressions.

Another effort to model fat-tailed loss distribution was undertaken by the so-called credit risk contagion models. The intuitive assumption is that a credit event at one company affects the solvency of related companies directly, thus increasing the concentration of loss events. Contagion models implement the framework of factor models adding contagion as changes to asset values of related companies as a result of defaults. The strength of a contagion is usually measured as a function of the percentage of sales from each company to each of the other companies. Unfortunately, this approach requires data that is seldom available. For details please refer to Egloff et al. (2004), Giesecke and Weber (2004).

A fundamentally different approach to credit risk modelling is provided by the dynamic intensity models. The theory underlying these models is math-

ematically complex and beyond the scope of this chapter. The basic idea is that an intensity process $\lambda(t)$ described by a stochastic differential equation exists. This process $\lambda(t)$ drives the default times. By definition default in these models happens if the corresponding default time is less than the planning horizon T. More details can be found in Duffie and Singleton (1997).

## 22.5 One Factor Models

The multiple factor model was introduced in (22.13). A one factor model simplifies the analysis since there is only one driving factor common to all obligors: $Y \sim N(0,1)$. In our discussion we concentrate on the KMV-Model. In a one factor setup we model the (standardised) log returns:

$$r_i = \sqrt{\omega}Y + \sqrt{1-\omega}Z_i \tag{22.23}$$

with idiosyncratic $Z_i \sim N(0,1)$. The uniform asset correlation is denoted $\omega$. As before $Z_i$ is assumed to be independent from the factor $Y$. Given a single factor and identical for all obligors $\omega$, we can rewrite equation (22.16) as:

$$p_i(y) = \Phi\left\{\frac{\Phi^{-1}(p_i) - \sqrt{\omega}y}{\sqrt{1-\omega}}\right\} \tag{22.24}$$

In order to demonstrate the dependence of $p_i(y)$ on the default probability given default $y$ values, let us fix $\omega = 20\%$ and $y \in \{-3, 0, 3\}$. The variable $y \sim N(0,1)$ can be interpreted as the state of the economy, $y = -3$ corresponds to a bad state, $y = 0$ means a typical state and $y = 3$ indicates a good state of the economy. See Figure 22.1. The joint default probability is given in the following proposition.

**Proposition 22.1**
*In a one-factor portfolio model with $L_i \sim B(1, p_i)$, $p_i$ from (22.24), the joint default probability (JDP) of two obligors is:*

$$JDP_{ij} = P(L_i = 1, L_j = 1) = \Phi_2\left\{\Phi^{-1}(p_i), \Phi^{-1}(p_j); \omega\right\},$$

*where $\Phi_2[\cdot, \cdot; \omega]$ denotes the bivariate normal cdf with correlation $\omega$.*

For a portfolio of $m$ obligors, the portfolio loss relative to the portfolio's total exposure is given by:

$$L^{(m)} = \sum_{i=1}^{m} w_i LGD_i L_i, \quad w_i = \frac{EAD_i}{\sum_{j=1}^{m} EAD_j}. \tag{22.25}$$

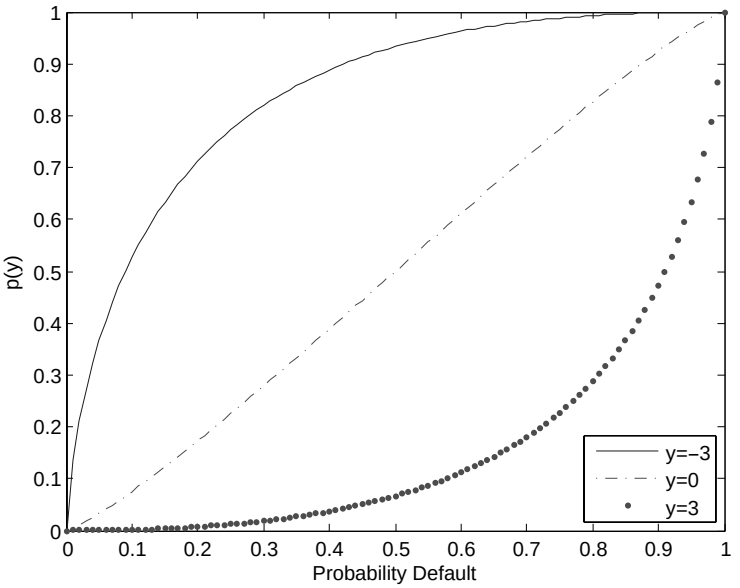


Figure 22.1: The default probability $p(y)$ as a function of the state of the economy.

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Bluhm et al. (2002) illustrate that for $m \to \infty$ the randomness of the portfolio loss $L^{(m)}$ solely depends on the randomness of the factor $Y$:

$$\mathrm{P}(lim_{m\to\infty}\{L^{(m)} - \mathsf{E}(L^{(m)}|Y)\} = 0) = 1. \tag{22.26}$$

Thus by increasing the number of obligors in the portfolio, the specific risk is completely removed and only the systematic risk arising from the volatility of the common factor remains in the portfolio.

Assuming uniform default probabilities $p_i$ for all obligors $i$ and applying KMV framework to our analysis we infer:

$$\mathsf{E}(L^{(m)}|Y) = \sum_{i=1}^{m} w_i \, \mathsf{E}(L_i|Y) = \Phi\left\{\frac{\Phi^{-1}(p_i) - \sqrt{\omega}y}{\sqrt{1-\omega}}\right\} =: p(Y),$$

so that from the formula (22.26) it follows that

$$L^{(m)} \overset{m\to\infty}{\longrightarrow} p(Y) = \Phi\left\{\frac{\Phi^{-1}(p_i) - \sqrt{\omega}y}{\sqrt{1-\omega}}\right\} \qquad \text{almost surely.} \tag{22.27}$$

We have established that for sufficiently large portfolios the percentage of defaulted loans given a certain state of economy $Y = y$ is approximately equal to the conditional default probability $p(Y)$. Now we want to derive the cdf of the limit loss variable $p(Y)$ and thus define the loss distribution. We denote here the limit of $L^{(m)}$ by $L$. For every $0 \leq x \leq 1$ we then have:

$$\begin{aligned} \mathrm{P}(L \leq x) &= \mathrm{P}(p(Y) \leq x) \qquad (22.28) \\ &= \mathrm{P}\left\{-Y \leq \frac{1}{\sqrt{\omega}}\left(\Phi^{-1}(x)\sqrt{1-\omega} - \Phi^{-1}(p)\right)\right\} \\ &= \Phi\left\{\frac{1}{\sqrt{\omega}}\left(\Phi^{-1}(x)\sqrt{1-\omega} - \Phi^{-1}(p)\right)\right\}. \end{aligned}$$

Thus we can now calculate the corresponding pdf and find such portfolio loss characteristics such as expected and unexpected losses. Obviously, we first need a factor model to define asset correlation $\omega$ and some market data to calibrate the default probability $p$.

## 22.6 Copulae and Loss Distributions

It could be shown that even holding the asset correlations and default probabilities fixed we can develop alternative models which lead to heavier-tailed loss distributions as compared to those induced by the assumptions of factor and sector models. In this section we will describe how to use copulae for constructing such loss distributions. For details on the concept of copulae please refer to chapter 17.

Copula presents an elegant way of understanding how a multivariate latent variable (e.g. asset returns) distribution determines the portfolio default distribution. From Sklar's Theorem we know that it is possible to extract copulae from known multivariate distribution functions or create new multivariate distributions by joining arbitrary marginal distributions in a known dependence structure, i.e. copula. In the factor models we assumed latent variables $r$ to have multivariate Gaussian distribution with correlation matrix $\Gamma$ that lead to the following copula of $r$:

$$C_{r,Ga}(u_1, ..., u_m) = \Phi_m\left\{\Phi_n^{-1}(u_1), ..., \Phi_n^{-1}(u_m)\right\} \qquad (22.29)$$

$C_{r,Ga}$ is a well-known Gaussian copula, which represents the dependence structure underlying $CreditMetrics^{TM}$ and KMV models. Replacing a normal dependency with a $t$-dependency will significantly shift mass into the tails of loss distribution as $t$-distributions possess fatter tails compared to

normal. Fatness of tails is highly sensitive to the chosen degrees of freedom, however the problem of calibrating copulae is beyond the scope of this book. Let $n \geq 3$ and $F_n$ be a $t$-distribution function with $n$ degrees of freedom. We denote the multivariate $t$-distribution function as $F_{n,\Gamma}$ where $\Gamma$ is the correlation matrix. The $t$-copulae is:

$$C_{n,\Gamma}(u_1, ..., u_m) = F_{n,\Gamma}\left\{F_n^{-1}(u_1), ..., F_n^{-1}(u_m)\right\} \tag{22.30}$$

where $u_1$,..., $u_m \in$ [0,1]. A new multivariate loss distribution can be created by combining the $t$-copulae $C_{n,\Gamma}$ with different marginal distributions. Below are a few examples of the two dimensional case.

- Bivariate Gaussian copulae with normal marginals. We generate pairs $(X_1, X_2)$ with:

  $$X_i = \sqrt{\omega}Y + \sqrt{1-\omega}Z_i$$

  where $Y$, $Z_1$, $Z_2 \sim$ N(0,1) i.i.d.
- Bivariate $t$-copulae with $t$-distributed margins. Generate $(X_1, X_2)$ with

  $$X_i = \sqrt{3}(\sqrt{\omega}Y + \sqrt{1-\omega}Z_i)/\sqrt{W}$$

  where $Y$, $Z_1$, $Z_2 \sim$ N(0,1) i.i.d. $W \sim \chi^2(3)$
- Bivariate $t$-copulae with normal margins. Generate $(X_1, X_2)$ with

  $$X_i = \Phi^{-1}[F_3\left\{\sqrt{3}(\sqrt{\omega}Y + \sqrt{1-\omega}Z_i)/\sqrt{W}\right\}]$$

  where $Y$, $Z_1$, $Z_2 \sim$ N(0,1) i.i.d. $W \sim \chi^2(3)$, $W$ independent of $Y$, $Z_1$, $Z_2$, and $F_3$ denoting the $t$-distribution function with 3 degrees of freedom.

In order to study the goodness-of-fit of different copulae on the default models we recall (22.23) and rewrite it in the form of a $t$-distributed log return:

$$r_i = \sqrt{n/W}\sqrt{\omega}Y + \sqrt{n/W}\sqrt{1-\omega}Z_i \sim t(n)$$

for $i = 1, ..., m$. The loss function is $L_i = \mathbf{1}\{\widetilde{r}_i < F_n^{-1}(p)\}$, and the default probability conditional on $Y$ and $W$ turns into:

$$\begin{aligned} p(y,w) &= P(r_i \leq F_n^{-1}(p)|Y = y, W = w) \\ &= P(\sqrt{n/W}\sqrt{\omega}Y + \sqrt{n/W}\sqrt{1-\omega}Z_i \leq F_n^{-1}(p)|Y = y, W = w) \\ &= \Phi\{\frac{\sqrt{w/n}F_n^{-1}(p) - \sqrt{\omega}y}{\sqrt{1-\omega}}\} \end{aligned} \tag{22.31}$$

Bluhm et al. (2002) simulated the portfolio loss in such a $t$-copulae model by looking at the distribution of (22.31) with independent $\{Y_i\}_{i=1}^N \sim N(0,1)$, $\{W_i\}_{i=1}^N \sim X^2(n)$, $N = 100000$, $n = 10, 40, 10000$. They concluded that for large $n$ the $t$-copulae resemble the Gaussian copulae. For small $n$ though there may be a big difference in the 99% quantiles.

To conclude this chapter we will show how to estimate asset corelation from historic default frequencies using one factor model. Our first step is to calibrate default probabilities. Table 22.1 presents Moody's historic corporate bond default frequencies from 1970 to 2004. For each rating class $R_i$ we calculate the mean and the standard error of the historic default frequencies. Then we use simple regression to fit the mean by an exponential function. As a result we can obtain fitted default probabilities $\mu_1, \ldots, \mu_6$ for all rating classes (see table 22.2), and we can then fit the volatilies of the default frequencies analogously.

The second step includes calculating the asset correlations. We refer to the formula (22.24) from the uniform factor model, in which we replace true default probability $p_i$ for the fitted mean default rate $\mu_i$. It can be shown that the following expression is true for the considered model:

$$\mathrm{Var}(\mathrm{P}(Y)) = \Phi_2\{\Phi^{-1}(p), \Phi^{-1}(p); \omega\} - p^2 \tag{22.32}$$

where we again replace the true unknown variance of default rate for the fitted default volatility $\sigma$. Thus, the asset correlation $\omega$ is the only unknown parameter in (22.32). The calibrated correlations are showed in the last column of table (22.2).

| Year | Aaa | Aa | A | Baa | Ba | B |
|---|---|---|---|---|---|---|
| 1970 | 0.00 | 0.00 | 0.00 | 0.27 | 4.12 | 20.78 |
| 1971 | 0.00 | 0.00 | 0.00 | 0.00 | 0.42 | 3.85 |
| 1972 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 7.14 |
| 1973 | 0.00 | 0.00 | 0.00 | 0.46 | 0.00 | 3.77 |
| 1974 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 10.00 |
| 1975 | 0.00 | 0.00 | 0.00 | 0.00 | 1.02 | 5.97 |
| 1976 | 0.00 | 0.00 | 0.00 | 0.00 | 1.01 | 0.00 |
| 1977 | 0.00 | 0.00 | 0.00 | 0.28 | 0.52 | 3.28 |
| 1978 | 0.00 | 0.00 | 0.00 | 0.00 | 1.08 | 5.41 |
| 1979 | 0.00 | 0.00 | 0.00 | 0.00 | 0.49 | 0.00 |
| 1980 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 4.94 |
| 1981 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 4.49 |
| 1982 | 0.00 | 0.00 | 0.26 | 0.31 | 2.72 | 2.41 |
| 1983 | 0.00 | 0.00 | 0.00 | 0.00 | 0.91 | 6.31 |
| 1984 | 0.00 | 0.00 | 0.00 | 0.36 | 0.83 | 6.72 |
| 1985 | 0.00 | 0.00 | 0.00 | 0.00 | 1.40 | 8.22 |
| 1986 | 0.00 | 0.00 | 0.00 | 1.33 | 2.03 | 11.73 |
| 1987 | 0.00 | 0.00 | 0.00 | 0.00 | 2.71 | 6.23 |
| 1988 | 0.00 | 0.00 | 0.00 | 0.00 | 1.24 | 6.36 |
| 1989 | 0.00 | 0.61 | 0.00 | 0.59 | 2.98 | 8.95 |
| 1990 | 0.00 | 0.00 | 0.00 | 0.00 | 3.35 | 16.18 |
| 1991 | 0.00 | 0.00 | 0.00 | 0.27 | 5.35 | 14.56 |
| 1992 | 0.00 | 0.00 | 0.00 | 0.00 | 0.30 | 9.03 |
| 1993 | 0.00 | 0.00 | 0.00 | 0.00 | 0.56 | 5.71 |
| 1994 | 0.00 | 0.00 | 0.00 | 0.00 | 0.24 | 3.82 |
| 1995 | 0.00 | 0.00 | 0.00 | 0.00 | 0.69 | 4.81 |
| 1996 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 1.44 |
| 1997 | 0.00 | 0.00 | 0.00 | 0.00 | 0.19 | 2.12 |
| 1998 | 0.00 | 0.00 | 0.00 | 0.12 | 0.63 | 4.26 |
| 1999 | 0.00 | 0.00 | 0.00 | 0.10 | 1.01 | 5.85 |
| 2000 | 0.00 | 0.00 | 0.00 | 0.38 | 0.89 | 5.49 |
| 2001 | 0.00 | 0.00 | 0.16 | 0.19 | 1.57 | 9.36 |
| 2002 | 0.00 | 0.00 | 0.16 | 1.21 | 1.54 | 4.97 |
| 2003 | 0.00 | 0.00 | 0.00 | 0.00 | 0.95 | 2.66 |
| 2004 | 0.00 | 0.00 | 0.00 | 0.00 | 0.19 | 0.65 |

Table 22.1: Moody's Corporate Bond Historic Default Frequency 1970-2004. All numbers are percentages values.

| Rating | Mean | Stand.Dev. | $\mu$ | $\sigma$ | $\rho$ |
|---|---|---|---|---|---|
| Aaa | 0.000% | 0.000% | 0.0124% | 0.1326% | 42% |
| Aa | 0.017% | 0.103% | 0.0413% | 0.2546% | 37% |
| A | 0.922% | 3.766% | 0.1374% | 0.4890% | 31% |
| Baa | 0.168% | 0.319% | 0.4565% | 0.9390% | 23% |
| Ba | 1.170% | 1.267% | 1.5171% | 1.8031% | 15% |
| B | 6.271% | 4.643% | 5.0417% | 3.4627% | 11% |
| Mean | 1.425% | 1.683% | 1.2011% | 1.1802% | 26% |

Table 22.2: Calibration Results due to exponential function fitting.

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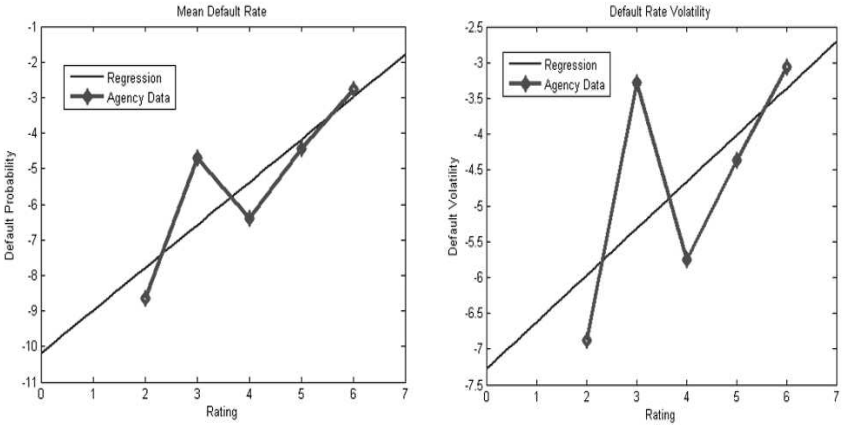


Figure 22.2: Mean default rate and default rate volatility. The red and blue lines represent the historic default and the regression by exponential function fitting correspondingly. All values are logged data.

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# A Technical Appendix

## A.1 Integration Theory

**Definition A.1**
*A* decomposition $\mathcal{Z}$ *of the interval* $[a,b]$ *is understood to be a set* $\mathcal{Z} \stackrel{\text{def}}{=} \{t_0, t_1, \ldots, t_n\}$ *of points* $t_j$ *with* $a = t_0 < t_1 < \ldots < t_n = b$. *Through this the interval* $[a,b]$ *is decomposed into* $n$ *sub-intervals* $[t_k, t_{k+1}]$, *where* $k = 0, 1, 2, \ldots, n-1$. $|\mathcal{Z}| \stackrel{\text{def}}{=} \max_k(t_{k+1} - t_k)$, *that is, the length of the largest resulting sub-interval and is referred to as the* refinement *of the decomposition* $\mathcal{Z}$.

**Definition A.2**
*For a function* $w : [a,b] \longrightarrow \mathbb{R}$ *and a decomposition* $\mathcal{Z} \stackrel{\text{def}}{=} \{t_0, t_1, \ldots, t_n\}$ *one defines the* variation *of* $w$ *with respect to* $\mathcal{Z}$ *as:*

$$V(\mathcal{Z}) \stackrel{\text{def}}{=} \sum_{k=0}^{n-1} |w(t_{k+1}) - w(t_k)|$$

$V \stackrel{\text{def}}{=} \sup_{\mathcal{Z}} V(\mathcal{Z})$ *is called the* total variation *of* $w$ *on* $[a,b]$. *If* $V < \infty$ *holds, then* $w$ *is of* finite variation *on* $[a,b]$.

**Theorem A.1**
*For a function* $w : [a,b] \longrightarrow \mathbb{R}$ *it holds that:*

1. $w$ *is of finite variation when* $w$ *is monotone,*
2. $w$ *is of finite variation when* $w$ *is Lipschitz continuous,*
3. $w$ *is bounded when* $w$ *is of finite variation.*

*Moreover, sums, differences and products of functions of finite variation are themselves of finite variation.*

**Definition A.3**
*Given the functions* $f, w : [a,b] \to \mathbb{R}$ *and a decomposition* $\mathcal{Z}$, *choose for*

*$k = 0, 1, \ldots, n-1$ partitions $\tau_k \in [t_k, t_{k+1}]$ and form:*

$$I(\mathcal{Z}, \boldsymbol{\tau}) \stackrel{\text{def}}{=} \sum_{k=0}^{n-1} f(\tau_k) \cdot \{w(t_{k+1}) - w(t_k)\}$$

*If $I(\mathcal{Z}, \boldsymbol{\tau})$ converges for $|\mathcal{Z}| \to 0$ to a limiting value $I$, which does not depend on the chosen decomposition $\mathcal{Z}$ nor on the choice of the partitions $\tau_k$, then $I$ is called the* **Riemann-Stieltjes integral** *of $f$. One writes:*

$$I = \int_a^b f(t) dw(t).$$

*For $w(t) = t$ we get the* **Riemann Integral** *as a special case of the Stieltjes Integrals.*

**Theorem A.2 (Characteristics of the Riemann-Stieltjes Integral)**

1. *If the corresponding integrals on the right hand side exist, then the* linearity characteristics *hold:*

$$\begin{aligned} \int_a^b (\alpha \cdot f + \beta \cdot g)\ dw &= \alpha \int_a^b f\ dw + \beta \int_a^b g\ dw \quad (\alpha, \beta \in \mathbb{R}) \\ \int_a^b f\ d(\alpha \cdot w + \beta \cdot v) &= \alpha \int_a^b f\ dw + \beta \int_a^b f\ dv \quad (\alpha, \beta \in \mathbb{R}) \end{aligned}$$

2. *If the integral $\int_a^b f dw$ and the integrals $\int_a^c f dw$ exist, then for $\int_c^b f dw$, $a < c < b$ it holds that:*

$$\int_a^b f dw = \int_a^c f dw + \int_c^b f dw$$

3. *If $f$ is continuous on $[a, b]$ and $w$ is of finite variation, then $\int_a^b f dw$ exists.*

4. *If $f$ is continuous on $[a, b]$ and $w$ is differentiable with a bounded derivative, then it holds that:*

$$\int_a^b f(t) dw(t) = \int_a^b f(t) \cdot w'(t) dt$$

5. Partial integration: *If* $\int_a^b f dg$ *or* $\int_a^b g df$ *exist, so does the other respective integral and it holds that:*

$$\int_a^b f dg + \int_a^b g df = f(b)g(b) - f(a)g(a)$$

6. *If* $w$ *is continuous, it holds that* $\int_a^b dw(t) = w(b) - w(a)$

7. *If* $f$ *is continuous on* $[a, b]$ *and* $w$ *is step-wise constant with discontinuity points* $\{c_k, k = 1, \ldots, m\}$*, then:*

$$\int_a^b f dw = \sum_{k=1}^{m} f(c_k) \cdot \left\{ w(c_k^+) - w(c_k^-) \right\}$$

*where* $c_k^+$ *(*$c_k^-$*) is the right (left) continuous limit and* $w(c_k^+) - w(c_k^-)$ *is the step height of* $w$ *on* $\{c_k\}$*.*

**Theorem A.3 (Radon-Nikodym)**
*Let* $\lambda$ *and* $\mu$ *be positive measures on* $(\Omega, \mathcal{F})$ *with*

1. $0 < \mu(\Omega) < \infty$ *and* $0 < \lambda(\Omega) < \infty$
2. $\lambda$ *is* absolutely continuous *with respect to* $\mu$*, then from* $\mu(A) = 0$ *it follows that* $\lambda(A) = 0$ *for all* $A \in \mathcal{F}$ *(written:* $\lambda \ll \mu$*).*

*When a non-negative* $\mathcal{F}$*-measurable function* $h$ *exists on* $\Omega$*, then it holds that:*

$$\forall A \in \mathcal{F}\text{: } \lambda(A) = \textstyle\int_A h d\mu;$$

*In particular, for all measurable functions* $f$ *it holds that:*

$$\int f d\lambda = \int f \cdot h \, d\mu.$$

**Remark A.1**
*One often uses the abbreviation* $\lambda = h \cdot \mu$ *in the Radon-Nikodym theorem and refers to* $h$ *as the* density *of* $\lambda$ *with respect to* $\mu$*. Due to its construction* $h$ *is also referred to as the* Radon-Nikodym derivative*. In this case one writes* $h = \frac{d\lambda}{d\mu}$*.*

An important tool in stochastic analysis is the transformation of measure, which is illustrated in the following example.

**Example A.1**
*Let $Z_1, \ldots, Z_n$ be independent random variables with standard normal distributions on the measurable space $(\Omega, \mathcal{F}, \mathrm{P})$ and $\mu_1, \ldots, \mu_n \in \mathbb{R}$. Then by*

$$\mathrm{Q}(d\omega) \stackrel{\text{def}}{=} \xi(\omega) \cdot \mathrm{P}(d\omega) \quad \textit{with} \quad \xi(\omega) \stackrel{\text{def}}{=} \exp\{\sum_{i=1}^{n} \mu_i Z_i(\omega) - \frac{1}{2}\mu_i^2\}$$

*an equivalent probability measure* Q *for* P *is defined. For the distribution of the $Z_1, \ldots, Z_n$ under the new measure* Q *it holds that:*

$$\begin{aligned}
& \mathrm{Q}(Z_1 \in dz_1, \ldots, Z_n \in dz_n) \\
= \; & \exp\{\sum_{i=1}^{n}(\mu_i z_i - \frac{1}{2}\mu_i^2)\} \cdot \mathrm{P}(Z_1 \in dz_1, \ldots, Z_n \in dz_n) \\
= \; & \exp\{\sum_{i=1}^{n}(\mu_i z_i - \frac{1}{2}\mu_i^2)\} \cdot (2\pi)^{-\frac{n}{2}} \exp\{-\frac{1}{2}\sum_{i=1}^{n} z_i^2\} dz_1 \ldots dz_n \\
= \; & (2\pi)^{-\frac{n}{2}} \exp\{-\frac{1}{2}\sum_{i=1}^{n}(z_i - \mu_i)^2\} dz_1 \ldots dz_n,
\end{aligned}$$

*in other words $Z_1, \ldots, Z_n$ are, with respect to* Q*, independent and normally distributed with expectations $\mathsf{E}_{\mathrm{Q}}(Z_i) = \mu_i$ and $\mathsf{E}_{\mathrm{Q}}[(Z_i - \mu_i)^2] = 1$. Thus the random variables $\widetilde{Z_i} \stackrel{\text{def}}{=} Z_i - \mu_i$ are independent random variables with standard normal distributions on the measurable space $(\Omega, \mathcal{F}, \mathrm{Q})$.*

Going from P to Q by multiplying by $\xi$ changes the expectations of the normally distributed random variables, but the volatility structure remains notably unaffected.

The following Girsanov theorem generalises this method for the continuous case, that is, it constructs for a given P-Brownian motion $W_t$ an equivalent measure Q and an appropriately adjusted process $W_t^*$, so that it represents a Q-Brownian motion. In doing so the ("arbitrarily" given) expectation $\mu_i$ is replaced by an ("arbitrarily" given) drift, that is, a stochastic process $X_t$.

**Theorem A.4 (Girsanov)**
*Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space, $W_t$ a Brownian motion with respect to* P*, $\mathcal{F}_t$ a filtration in $\mathcal{F}$ and $X_t$ an adapted stochastic process. Then*

$$\xi_t \stackrel{\text{def}}{=} \exp\left(\int_0^t X_u dW_u - \frac{1}{2}\int_0^t X_u^2 du\right)$$

*defines a martingal with respect to* P *and $\mathcal{F}_t$. The process $W_t^*$ defined by*

$$W_t^* \stackrel{\text{def}}{=} W_t - \int_0^t X_u du$$

*is a Wiener process with respect to the filtration* $\mathcal{F}_t$ *and*

$$\mathrm{Q} \stackrel{\text{def}}{=} \xi_T \cdot \mathrm{P} \tag{A.1}$$

*is a* P *equivalent probability measure* Q*.*

The Girsanov theorem thus shows that for a P-Brownian motion $W_t$ an equivalent probability measure Q can be found such that $W_t^*$, as a Q-Brownian motion at time $t$, contains the drift $X_t$. In doing so (A.1) means that: $\int_\Omega \mathbf{1}(\omega \in A) d\mathrm{Q}(\omega) = \mathrm{Q}(A) \stackrel{\text{def}}{=} \int_\Omega \mathbf{1}(\omega \in A)\xi_T d\mathrm{P}(\omega) = \mathsf{E}_\mathrm{P}[\mathbf{1}(\omega \in A)\xi_T]$ for all $A \in \mathcal{F}$.

**Remark A.2**
*With the relationships mentioned above* $\xi_t$ *is by all means a martingale with respect to* P *and* $\mathcal{F}_t$ *when the so-called* Novikov Condition

$$\mathsf{E}_\mathrm{P}\Big[\exp(\int_0^t X_u^2 du)\Big] < \infty \quad \textit{for all } t \in [0,T]$$

*is met, that is, when* $X_t$ *does not vary too much.*

Another important tool used to derive the Black-Scholes formula by means of martingale theory is the martingale representation theory. It states that every Q-martingale under certain assumptions can be represented by a predetermined Q-martingale by means of a square-integrable process.

**Theorem A.5 (Martingale Representation theorem)**
*Let* $M_t$ *be a martingale with respect to the probability measure* Q *and the filtration* $\mathcal{F}_t$*, for which the volatility process* $\sigma_t$ *of* Q *almost surely* $\sigma_t \neq 0$ *for all* $t \in [0,T]$*, where* $\sigma_t^2 = \mathsf{E}_\mathrm{Q}[M_t^2|\mathcal{F}_t]$*. If* $N_t$ *is another martingale with respect to* Q *and* $\mathcal{F}_t$*, then (uniquely defined) on* $\mathcal{F}_t$ *an adapted stochastic process* $H_t$ *exists with* $\int_0^T H_t^2\sigma_t^2 dt < \infty$ *with:*

$$N_t = N_0 + \int_0^t H_s dM_s.$$

**Example A.2**
*It is easy to show that the standard Wiener process* $W_t$ *with respect to the probability measure* P *is a martingale with respect to* P *and its corresponding filtration* $\mathcal{F}_t$*. If* $X_t$ *is another martingale with respect to* P *and* $\mathcal{F}_t$*, then according to the previous theorem a* $\mathcal{F}_t$ *adapted stochastic process* $H_t$ *exists, so that*

$$X_t = X_0 + \int_0^t H_s dW_s.$$

**Remark A.3**
*Writing the last expression in terms of derivatives:*

$$dX_t = H_t dW_t.$$

*The example shows once again that a martingale cannot possess a drift.*

## A.2 Portfolio Strategies

The portfolio of an investor at time $t$, i.e., the market value of the single equities (contracts) in his portfolio at time $t$, is dependent on the development of the price $\{\boldsymbol{S}_s; s < t\}$, $\boldsymbol{S}_s = (S_s^1, \ldots, S_s^d)^\top \in \mathbb{R}^d$ up to time $t$, that is, on the information that is available at that particular time point. Given this, it is obvious that his strategy, i.e., the development of his portfolio's value over time, should be modelled as a $\mathcal{F}_t$ adapted $d$-dimensional stochastic process $\phi_t$. In doing so $\phi_t^i(\omega)$ represents how much in state $\omega$ of the security $i$ is in his portfolio at time $t$ , where negative values indicate a short sell of the corresponding contract.

**Definition A.4**
*Assume the following market model: $\mathcal{M} = (\Omega, \mathcal{F}, \mathrm{P}, \mathcal{F}_t, \boldsymbol{S}_t)$. A $d$-dimensional stochastic process $\phi_t$ adapted on the filtration $\mathcal{F}_t$ is called a portfolio strategy. The stochastic process $V(\phi_t)$ with $V(\phi_t) \stackrel{\text{def}}{=} \sum_{i=1}^{d} \phi_t^i S_t^i$ is called the value of the strategy $\phi$.*

**Example A.3**
*In the Black-Scholes model two financial instruments are traded on the market: a risky security $S$ (stock) and a riskless security $B$ (zero bond). As in Chapter 5, the stock price $S_t$ is assumed to follow a geometric Brownian motion, so that the following stochastic differential equation is satisfied:*

$$dS_t = S_t(\mu dt + \sigma dW_t) \tag{A.2}$$

*The price of the zero bond $B_t$ satisfies the differential equation:*

$$dB_t = rB_t dt$$

*with a constant $r$. Without loss of generality it can be assumed that $B_0 = 1$, which leads to $B_t = exp(rt)$.*

*The corresponding market model is thus $\mathcal{M}_{BS} = (\Omega, \mathcal{F}, \mathrm{P}, \mathcal{F}_t, \boldsymbol{S}_t)$, where $\boldsymbol{S}_t \stackrel{\text{def}}{=} (S_t, B_t)^\top \in \mathbb{R}^2$.*

*The two-dimensional stochastic process* $\phi_t = (a_t, b_t)^\top$ *now describes a portfolio strategy in which* $a_t(\omega)$ *gives the number of stocks and* $b_t(\omega)$ *gives the number of bonds in the portfolio at time* $t$ *in state* $\omega$*. The value of the portfolio at time* $t$ *is then a random variable*

$$V(\phi_t) = a_t S_t + b_t B_t.$$

A particularly important portfolio strategy is that once it is implemented it does not result in any cash flows over time, i.e., when the portfolio is re-balanced no payments are necessary. This means that eventual income (through selling securities, receiving dividends, etc.) is exactly offset by required payments (through buying additional securities, transaction costs, etc.) This is referred to as a self-financing strategy. One gets the impression that the change in value of the portfolio only occurs as the price of the participating securities changes.

**Definition A.5**
*Let* $\mathcal{M} = (\Omega, \mathcal{F}, \mathrm{P}, \mathcal{F}_t, \boldsymbol{S}_t)$ *be a market model and* $\phi$ *a portfolio strategy with the value* $V(\phi_t)$*. Then* $\phi$ *is called*

1. self-financing*, when* $dV(\phi_t) = \sum_{i=1}^{d} \phi_t^i dS_t^i$ *holds (*P*-a.s.),*
2. admissible*, when* $V(\phi_t) \geq 0$ *holds (*P*-a.s.).*

Below the Black-Scholes model will be considered. The subsequent specification shows that arbitrage is not possible in such a market: There is no admissible self-financing strategy with a starting value of $V(\boldsymbol{\phi}_0) = 0$, whose end value $V(\boldsymbol{\phi}_T)$ is positive with a positive probability.

**Lemma A.1**
*In the Black-Scholes model* $\mathcal{M}_{BS} = (\Omega, \mathcal{F}, \mathrm{P}, \mathcal{F}_t, \boldsymbol{S}_t), \boldsymbol{S}_t = (S_t, B_t)^\top$*, the portfolio strategy* $\phi_t = (a_t, b_t)^\top$ *is exactly self-financing when the discounted process* $\tilde{V}_t$ *with* $\tilde{V}_t = e^{-rt} V_t$ *satisfies the stochastic differential equation*

$$d\tilde{V}_t = a_t d\tilde{S}_t,$$

*where* $\tilde{S}_t = e^{-rt} S_t$ *describes the discounted stock price.*

The explicit specification of the corresponding strategy can be left out when it is clear from the context and we write $V_t = V(\phi_t)$. With the help of the Girsanov theorem a P equivalent measure Q can be constructed, under which

the process of the discounted stock prices is a martingale. Using (A.2) one obtains

$$d\tilde{S}_t = \tilde{S}_t\{(\mu - r)dt + \sigma dW_t\}. \tag{A.3}$$

By setting

$$X_t \stackrel{\text{def}}{=} -\frac{\mu - r}{\sigma}$$

the Novikov condition (see Remark A.2) is obviously fulfilled. Therefore, for Q with

$$\begin{aligned} \frac{d\mathsf{Q}}{d\mathsf{P}} = \xi_T &= \exp(\int_0^T X_u dW_u - \frac{1}{2}\int_0^T X_u^2 du) \\ &= \exp\{-\frac{\mu - r}{\sigma}W_T - \frac{1}{2}(\frac{\mu - r}{\sigma})^2 T\} \end{aligned}$$

$W_t^* \stackrel{\text{def}}{=} W_t + \frac{\mu - r}{\sigma}t$ is a Q-Brownian Motion according to the Girsanov theorem. Because of (A.3) and using the definition of $W_t^*$ it holds that

$$d\tilde{S}_t = \tilde{S}_t \sigma dW_t^*. \tag{A.4}$$

According to Itô's lemma this becomes

$$\tilde{S}_t = \tilde{S}_0 \exp\left(\int_0^t \sigma dW_u^* - \frac{1}{2}\int_0^t \sigma^2 du\right)$$

and solves the stochastic differential equation. Since $\sigma$ is constant, for all $t$ the Novikov condition holds

$$\mathsf{E}\left[\exp\left(\int_0^t \sigma^2 du\right)\right] < \infty.$$

According to Remark A.2

$$\exp\left(\int_0^t \sigma dW_u^* - \frac{1}{2}\int_0^t \sigma^2 du\right),$$

that is $\tilde{S}_t$, is also a Q-martingale.

Q represents with respect to $\widetilde{S}_t$ a P equivalent martingale measure. It can be shown that given this form, it can be uniquely determined.

From the Definition of $W_t^*$ and with the help of (A.2) one obtains

$$dS_t = S_t(rdt + \sigma dW_t^*),$$

i.e., under the measure Q the expected value of the risky securities is equivalent to the certain value of the riskless bonds. Because of this the martingale

measure Q is also called the *risk neutral* probability measure and contrary to this P is called the *objective* or *physical* probability measure of the Black-Scholes markets.

As a result of the Q-martingale properties of $\widetilde{S}_t$, due to Lemma A.1, the discounted value of a self-financing strategy $\widetilde{V}_t$ is itself a Q-martingale. Consequently it holds that: If the starting value of an admissible self-financing strategy is equal to zero, then its value at all later time points $t$ must also be equal to zero. Thus in using an admissible self-financing strategy, there is no riskless profit to be made: The Black-Scholes market is free of arbitrage.

The following theorem represents the most important tool used to value European options with the help of the Black-Scholes model. It secures the existence of an admissible self-financing strategy that duplicates the option, thus the value of which can be calculated using martingale theory.

**Theorem A.6**
*Assume that the Black-Scholes model $\mathcal{M}_{BS}$ is given. The function $X$ describes the value of an European option at the time to maturity $T$ and is* Q*-integrable.*

*a) Then an admissible self-financing strategy $(a_t, b_t)^\top$ exists, which duplicates $X$ and whose value $V_t$ for all $t$ is given by*

$$V_t = \mathsf{E}_{\mathrm{Q}}[e^{-r(T-t)}X \mid \mathcal{F}_t]. \tag{A.5}$$

*b) If the value $V_t$ in a) is dependent on $t$ and $S_t$ and is written as a function $V_t = F(t, S_t)$ with a smooth function $F$, then it holds for the corresponding strategy that*

$$a_t = \frac{\partial F(t, S_t)}{\partial S_t}.$$

**Proof:**

1. One defines $V_t$ by (A.5), where the function defined follows from the Q-integrability of $X$. Due to

$$\widetilde{V}_t = e^{-rt}V_t = \mathsf{E}_{\mathrm{Q}}[e^{-rT}X \mid \mathcal{F}_t]$$

one identifies $\widetilde{V}_t$ as Q-martingale. One should notice that $e^{-rT}X$, exactly like $X$, is only dependent on the state at date $T$ and thus it can be classified as a random variable on $(\Omega, \mathcal{F}_t, \mathrm{Q})$.

$\mathcal{F}_t$ represents, at the same time, the natural filtration for the process $W^*$, which, as was seen above, is also a Q-martingale. Therefore, according to Theorem A.5 using the martingale representation, a process $H_t$ exists, adapted on $\mathcal{F}_t$ with $\int_0^T H_t^2\sigma^2 dt < \infty$ Q-almost sure, so that for all $t$ it holds that:

$$\widetilde{V}_t = \widetilde{V}_0 + \int_0^t H_s dW_s^* = V_0 + \int_0^t H_s dW_s^*.$$

Thus one sets:

$$a_t \stackrel{\text{def}}{=} \frac{H_t}{\sigma \cdot \widetilde{S}_t} \quad , \quad b_t \stackrel{\text{def}}{=} \widetilde{V}_t - a_t\widetilde{S}_t.$$

Then after a simple calculation it holds that:

$$a_t S_t + b_t B_t = V_t$$

and $(a_t, b_t)^\top$ is a $X$ duplicating strategy. Furthermore, with (A.4) it holds for all $t$:

$$a_t d\widetilde{S}_t = a_t \widetilde{S}_t \sigma dW_t^* = H_t dW_t^* = d\widetilde{V}_t,$$

i.e., $(a_t, b_t)^\top$ is according to Lemma A.1 self-financing. Due to the non-negativity of $X$ and the definition of $V_t$, $(a_t, b_t)^\top$ is also admissible.

2. For $V_t = F(t, S_t)$ it holds using Itô's lemma:

$$\begin{aligned} d\widetilde{V}_t &= d\{e^{-rt}F(t,S_t)\} \\ &= \frac{\partial\{e^{-rt}F(t,S_t)\}}{\partial S_t} dS_t + A(t,S_t)dt \\ &= \frac{\partial F(t,S_t)}{\partial S_t} e^{-rt} S_t(rdt + \sigma dW_t^*) + A(t,S_t)dt \\ &= \frac{\partial F(t,S_t)}{\partial S_t} \widetilde{S}_t \sigma dW_t^* + \widetilde{A}(t,S_t)dt \\ &= \frac{\partial F(t,S_t)}{\partial S_t} d\widetilde{S}_t + \widetilde{A}(t,S_t)dt. \end{aligned}$$

Since not only $\widetilde{V}_t$ but also $\widetilde{S}_t$ are Q-martingales, the drift term $\widetilde{A}(t, S_t)$ must disappear. According to part a) of the theorem the corresponding strategy is self-financing and thus using Lemma A.1 the claim follows.

□

**Remark A.4** *With the relationships of the preceding theorems, $V_t$ is called the fair price for option $X$ at date $t$, because at this price, according to the*

*previous arguments, there is no arbitrage possible for either the buyer or the seller of the option. Equation (A.5) is called the risk neutral valuation formula, since it gives the fair price of the option as the (conditional) expectation of the (discounted) option value at maturity with respect to the risk neutral measure of the Black-Scholes model.*

The result obtained from the last theorem has already been formulated in Chapter 6 as equation (6.24).

**Corollary A.1**
*The relationships of the preceding theorems hold. If the value $X$ of the European option at date $T$ is a function $X = f(S_T)$ dependent on the stock price $S_T$, then it holds that $V_t = F(t, S_t)$, where $F$ for $x \in [0, \infty[$ and $t \in [0, T]$ is defined by:*

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{+\infty} f\left\{ x e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma y\sqrt{T-t}} \right\} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \ . \qquad \text{(A.6)}$$

**Proof:**
With respect to Q, $S_t$ contains the drift $r$ and thus it holds that

$$S_t = S_0 \exp\{(r - \frac{\sigma^2}{2})t + \sigma W_t^*\}.$$

Thus $S_T$ can be written in the following form:

$$S_T = S_t(S_T S_t^{-1}) = S_t \exp\{(r - \frac{\sigma^2}{2})(T-t) + \sigma(W_T^* - W_t^*)\}.$$

Since $S_t$ is measurable with respect to $\mathcal{F}_t$ and $W_T^* - W_t^*$ is independent of $\mathcal{F}_t$, one obtains

$$\begin{aligned} V_t &= \mathsf{E}_{\mathrm{Q}}[e^{-r(T-t)} f(S_T) \mid \mathcal{F}_t] \\ &= \mathsf{E}_{\mathrm{Q}}\left[ e^{-r(T-t)} f(S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(W_T^*-W_t^*)}) \mid \mathcal{F}_t \right] \\ &= \mathsf{E}_{\mathrm{Q}}\left[ e^{-r(T-t)} f(x e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(W_T^*-W_t^*)}) \right]_{x=S_t} \end{aligned}$$

From this it can be calculated that $V_t = F(t, S_t)$. □

**Example A.4**
*Consider a European call $X = \max\{0, S_T - K\}$. Using (A.6) the value at date $t$ is exactly the value given by the Black-Scholes formula in Chapter 6.*

$$C(t, S_t) \stackrel{\text{def}}{=} V_t = S_T \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2)$$

*with*

$$d_1 \stackrel{\text{def}}{=} \frac{\ln(\frac{S_T}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 \stackrel{\text{def}}{=} \frac{\ln(\frac{S_T}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}.$$

# Frequently Used Notations

$x \stackrel{\text{def}}{=} \ldots$ $x$ is defined as ...
$\mathbb{R}$ real numbers
$\overline{\mathbb{R}} \stackrel{\text{def}}{=} \mathbb{R} \cup \{\infty, \infty\}$
$A^\top$ transpose of matrix $A$
$X \sim D$ the random variable $X$ has distribution $D$
$\mathsf{E}[X]$ expected value of random variable $X$
$Var(X)$ variance of random variable $X$
$\mathrm{Cov}(X, Y)$ covariance of two random variables $X$ and $Y$
$\mathrm{N}(\mu, \Sigma)$ normal distribution with expectation $\mu$ and covariance matrix $\Sigma$, a similar notation is used if $\Sigma$ is the correlation matrix
$\Phi$ standard normal cummulative distribution function
$\varphi$ standard normal density function
$\chi^2_p$ chi-squared distribution with $p$ degrees of freedom
$t_p$ $t$-distribution (Student's) with $p$ degrees of freedom
$W_t$ Wiener process
$\mathrm{P}[A]$ or $\mathrm{P}(A)$ probability of a set $A$
$\mathbf{1}$ indicator function
$(F \circ G)(x) \stackrel{\text{def}}{=} F\{G(x)\}$ for functions $F$ and $G$
$x \approx y$ $x$ is approximately equal to $y$
$\alpha_n = \mathcal{O}(\beta_n)$ iff $\frac{\alpha_n}{\beta_n} \longrightarrow$ constant, as $n \longrightarrow \infty$
$\alpha_n = o(\beta_n)$ iff $\frac{\alpha_n}{\beta_n} \longrightarrow 0$, as $n \longrightarrow \infty$
$\mathcal{F}_t$ is the information set generated by all information available at time $t$
Let $A_n$ and $B_n$ be sequences of random variables.
$A_n = \mathcal{O}_p(B_n)$ iff $\forall \varepsilon > 0\ \exists M,\ \exists N$ such that $\mathrm{P}[|A_n/B_n| > M] < \varepsilon,\ \forall n > N$.
$A_n = o_p(B_n)$ iff $\forall \varepsilon > 0\ :\ \lim_{n\to\infty} \mathrm{P}[|A_n/B_n| > \varepsilon] = 0$.

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